PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

A Modification of Oseen's Approximate Equation for the Motion in Two Dimensions of a Viscous Incompressible Fluid

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A Modification of Oseen's *Approximate Equation for the Motion in Two Dimensions of a Viscous Incompressible Fluid.*

By R. V. Southwell, *F.R.S.*, *and* H. B. Squire.

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[Plate 1.]

Part I.—General Considerations.

1. This paper is concerned with the steady motion in two dimensions, past a fixed cylindrical body, of an incompressible fluid possessing finite viscosity. Alternatively, the fluid may be imagined to be stationary at infinity, and the cylinder to move through it with uniform velocity in a direction normal to its own axis. The problem has importance for aeronautics, since a solution would permit the calculation of what in modern aerofoil theory is termed the "profile drag."*

2. Treating the cylinder as fixed, we take rectangular axes Ox, Oy, Oz, of which Oz is along its axis and Ox is the direction of the undisturbed stream (*i.e.*, of the flow at infinity). The motion of the fluid is then defined by component velocities u, v in the directions Ox, Oy respectively. The third component w vanishes in the two-dimensional motion, and the condition of incompressibility requires that u and v shall be related by the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (1)$$

In addition to (1), u and v must satisfy the equations†

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \mathrm{X} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \mathrm{Y} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v, \end{aligned}\right\} \qquad (2)$$

in which ρ is the density and ν the "kinematic viscosity" of the fluid, p stands for the "mean pressure," and X and Y are the components of the extraneous forces, per unit mass, at a point (x, y). They are also subject to the conditions

$$u = v = 0, \qquad (3)$$

* H. Glauert, "Aerofoil and Airscrew Theory," § 9.5.

† Lamb, "Hydrodynamics" (4th ed.), § 328.

at all points on the boundary of the cylinder; and to the condition

$$u = \text{const.} = \mathrm{U} \text{ (say)}, \quad v = 0, \qquad (4)$$

at an infinite distance from the cylinder in any direction.

3. The solution of our problem turns on the finding of an expression for the "vorticity" ζ, defined by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \qquad (5)$$

For on elimination of p from (2) by cross-differentiation, and making use of (1), we obtain the equation

$$u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \frac{\partial \mathrm{Y}}{\partial x} - \frac{\partial \mathrm{X}}{\partial y} + \nu\nabla^2\zeta, \qquad (6)$$

in which, as in equation (2),

$$\nabla^2 \text{ denotes the operator } \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad (7)$$

Moreover, the relation (1) permits the introduction of a "stream function" ψ, related with u and v by the equations

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \qquad (8)$$

and then we have from (5) the expression

$$\zeta = \nabla^2\psi, \qquad (9)$$

and from (3) and (4) the boundary conditions

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} &= 0, \quad \text{on the cylindrical boundary,} \\ \frac{\partial \psi}{\partial x} = 0, \quad \frac{\partial \psi}{\partial y} &= -\mathrm{U}, \text{ at infinity.} \end{aligned}\right\} \qquad (10)$$

When, as in the problem now under discussion, the body forces X, Y are zero or conservative, equation (6) takes the simpler form

$$u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu\nabla^2\zeta. \qquad (11)$$

This, when we substitute for u, v, ζ from (8) and (9), becomes an equation in ψ only, and our problem thus reduces to the finding of a form for ψ which shall satisfy equation (11) together with the boundary conditions (10). But, as is well known, the difficulties presented by the non-linear equation in ψ have so far proved insuperable, and no complete and exact solution of our problem exists (that is, for fluids possessing finite viscosity). If the motion is sufficiently slow, the second-order terms may be neglected in comparison with the others: something can be done with the first-order equation which is then presented, but (as STOKES pointed out*) it proves to be impossible to satisfy all the conditions. At very high speeds, the right-hand side of (11) is negligible except in a very narrow region comprising the "boundary layer" and the "wake": in these circumstances we can proceed on lines which have been developed by PRANDTL and his school.†

OSEEN'S *Approximation.*

4. With the aim of rendering the governing equation tractable in cases where neither the right-hand side nor the second-order terms of (11) can be neglected, OSEEN‡ has proposed to substitute for u and v, on the left-hand side, the "*undisturbed*" velocities of the fluid. In the problem now under consideration, these undisturbed velocities are given by (4), and OSEEN'S procedure yields as the modified form of the governing equation

$$U \frac{\partial \zeta}{\partial x} = \nu \nabla^2 \zeta. \qquad (12)$$

Equation (12) is linear and accordingly tractable; in the hands of Sir HORACE LAMB it has yielded a result for the circular cylinder which has been stated to agree well with experimental determinations of drag at low values of REYNOLDS' number (LAMB, *loc. cit.*, § 343). OSEEN'S method of approximation is not restricted to motion in two dimensions, and it has been applied with success to the sphere and to the spheroid.

Alternative Method of Approximation.

5. In this paper we investigate a procedure alternative to OSEEN'S, also leading to a differential equation linear in ζ (or ψ), which on physical grounds would seem to offer hopes of a closer approximation.§ It is evident that OSEEN'S substitution, though valid at points very far distant from the cylinder, is open to the objection that at points less distant no account is taken of the share which the velocity component v takes in determining the convection of vorticity; OSEEN'S equation (12) represents this convection as being conditioned by the "*undisturbed velocities*" U, 0, whereas actually it is conditioned by the "*disturbed velocities*" u, v. Now we know from experiment that the disturbed velocities u, v are approximately irrotational in parts of the field

* LAMB, *loc. cit.*, § 343.

† GLAUERT, *loc. cit.*

‡ 'Ark. Mat. Astr. Fys,' vol. 6, No. 29 (1910). The references in this paper are to a full account given by LAMB, *loc. cit.*, §§ 340–343.

§ In the Summary (p. 62) references are given to some earlier investigations on similar lines.

which are not very near to the solid boundary or to the "wake." Hence, to replace them by irrotational velocities should be a closer approximation to the truth than to replace them by undisturbed velocities.

Accordingly we substitute for u and v, in (11), the expressions

$$u = -\frac{\partial \alpha}{\partial x}, \quad v = -\frac{\partial \alpha}{\partial y}, \quad \ldots \ldots \quad (13)$$

in which α is some plane harmonic function, at present unspecified. We write the resulting equation in the form

$$\left[\frac{\partial \alpha}{\partial x} + \nu \frac{\partial}{\partial x}\right]\frac{\partial \zeta}{\partial x} + \left[\frac{\partial \alpha}{\partial y} + \nu \frac{\partial}{\partial y}\right]\frac{\partial \zeta}{\partial y} = 0, \quad \ldots \ldots \quad (14)$$

and this, after multiplying through by

$$\chi = e^{\alpha/\nu}, \quad \ldots \ldots \quad (15)$$

we throw into the equivalent form

$$\left[\frac{\partial}{\partial x}\left(\chi \frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi \frac{\partial}{\partial y}\right)\right]\zeta = 0, \quad \ldots \ldots \quad (16)$$

in which, as before,

$$\zeta = \nabla^2 \psi. \quad \ldots \ldots \quad (9)\ bis.$$

Remarks on the new form of approximate equation.

6. Equation (16) is our alternative to OSEEN's approximate equation (12). Like that equation, it is a differential equation linear in ζ (χ being regarded as a specified function of x and y) and accordingly tractable. The form of the solution will depend upon the absolute magnitude of the plane harmonic function α,—that is, upon the speed of the irrotational motion to which the actual motion must approximate; hence our solution will offer some account of the changes in the flow-pattern which occur with increasing "REYNOLDS' number."

At low values of REYNOLDS' number (*i.e.*, when ν is large, or when α is very nearly constant, so that the irrotational velocities (13) are small), equation (14) approximates to the "slow motion equation"

$$\nabla^2 \zeta = 0. \quad \ldots \ldots \quad (17)$$

At the other end of the scale (when ν is small, or when the irrotational velocities are large), equation (14)—which is equivalent to (16)—tends to the limiting form

$$\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \zeta}{\partial x} + \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \zeta}{\partial y} = 0, \quad \ldots \ldots \quad (18)$$

which expresses the condition that ζ is independent of α. Thus, in the solution of our equation which is applicable to high value of REYNOLDS' number, the vorticity ζ will be constant along a line of constant β, where β is the plane-harmonic function conjugate to α; and since a line of constant β will be a stream-line for the irrotational velocities defined by (13), the solution will satisfy the requirements of our problem, to the extent that the stream-lines for these irrotational velocities conform with the stream-lines of the actual flow.

We may conclude that (16) will be a satisfactory approximation to the exact form (11) of the governing equation throughout the whole of the "speed-scale" range, provided that α is the velocity potential function appropriate to a cylinder of the form which we are considering. For the experimental evidence (§ 5) indicates that the actual flow pattern at high speeds does in fact approximate to the irrotational flow pattern, except at points very close to the boundary of the cylinder and in the "wake."

7. Equation (16) can be interpreted, physically, in a number of different ways. For example, ζ may be taken to be the transverse displacement of a flexible membrane having a tension (the same in all directions, but varying from point to point) which is given by χ. More important (because it might be made the basis of an experimental method of solving our approximate equation*) is the electrical analogue, in which ζ stands for the potential in a thin plate of material having variable conductivity χ. In our hydrodynamic application the form of χ is of course limited by (15), in which α is a certain plane-harmonic function; in the electrical analogue no restriction is imposed on χ, other than by the requirement that it shall be single-valued.

Mathematical Aspects.

8. We turn now to the mathematical aspects of equation (16), and examine the consequences of a change of variables, from x and y to α and its associated plane-harmonic function β. Since

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}, \quad \frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x}, \qquad (19)$$

and writing

$$h^2 = \left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2, \qquad (20)$$

we have

$$\left.\begin{aligned} \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv h^2 \left[\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right] \\ \frac{\partial \alpha}{\partial x} \cdot \frac{\partial}{\partial x} + \frac{\partial \alpha}{\partial y} \cdot \frac{\partial}{\partial y} \equiv h^2 \frac{\partial}{\partial \alpha}, \end{aligned}\right\} \qquad (21)$$

and

whence it follows that (14) may be written in the form†

$$\left[\frac{\partial}{\partial \alpha} + \nu \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right)\right] \zeta = 0. \qquad (22)$$

* *Cf.* the investigations of C. F. SHARMAN and G. I. TAYLOR relating to compressible fluids, 'Proc. Roy. Soc.,' A, vol. 121, p. 194 (1928).

† In this form the equation illustrates clearly the remarks made in § 6. When ν is large, ζ satisfies very approximately the slow-motion equation (17): when ν is small, we have $\frac{\partial \zeta}{\partial \alpha} \doteqdot 0$.

Multiplying as before by

$$\chi = e^{\alpha/\nu}, \qquad \text{(15) } bis$$

we have the equation

$$\left[\frac{\partial}{\partial\alpha}\left(\chi\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\chi\frac{\partial}{\partial\beta}\right)\right]\zeta = 0, \qquad (23)$$

which shows that the form of (16) has not been altered by the change of variables. On the other hand, substitution from (21) in (9) yields the equation

$$\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right]\psi = \frac{\zeta}{h^2}, \qquad (24)$$

where h^2 is given by (20).

Equations (23) and (24) may be regarded as the governing equations of our problem, replacing (16) and (9). The conditions at the boundary of the solid cylinder, namely,

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0,$$

are replaced by

$$\frac{\partial\psi}{\partial\alpha} = \frac{\partial\psi}{\partial\beta} = 0, \qquad (25)$$

and if we assume (as in § 6) that α, β are respectively the velocity potential and stream function for irrotational flow past the cylinder in question, so that (for symmetrical profiles) the contour ($\beta = 0$) represents the profile and its line of symmetry, equations (25) may be replaced by the conditions

$$\left.\begin{array}{l}\psi = 0, \text{ when } \beta = 0, \\ \dfrac{\partial\psi}{\partial\beta} = 0, \text{ on that part of the line } (\beta = 0) \text{ which} \\ \qquad \text{represents the profile of our cylinder.}\end{array}\right\} \qquad (26)$$

The condition at infinity is, of course,

$$\psi = \beta. \qquad (27)$$

Part II.—Solution of the Approximate Equation for the case of Flow past a Thin Plate.

Formulation of the Problem.

9. Our attack on the problem presented by equations (23), (24), (26) and (27) may be illustrated most simply in relation to a cylinder having the form of a flat plate, and lying with its plane in the direction of the undisturbed stream. The length of the plate (in the direction Oz, § 2) is infinite, but its breadth (in the direction Ox) is finite, and it has zero thickness. This particular example of our problem has been treated (on purely analytical lines) in a joint paper by L. Bairstow, B. M. Cave and E. D. Lang.*

* '*Phil. Trans.*,' A., vol. 223, p. 383 (1923).

The irrotational flow has uniform velocity (U, say) in the direction Ox, and no component velocity in the direction Oy. Thus in (13) and (19) we must take

$$\alpha = -Ux, \quad \beta = -Uy,$$

so that

$$h^2 = U^2, \qquad (28)$$

by (20). The plane of the plate may be taken to coincide with the stream-line ($\beta = 0$), and its breadth (or length L up-stream) to occupy the part of the line ($\beta = 0$) lying between the equipotentials ($\alpha = \alpha_1$) and ($\alpha = \alpha_2$), where

$$\alpha_1 - \alpha_2 = UL.$$

The "REYNOLDS' number" of the motion considered will then be given by

$$R = \frac{UL}{\nu} = \frac{\alpha_1 - \alpha_2}{\nu}. \qquad (29)$$

Everything being symmetrical with respect to the (x, y) plane, it is evident that the component velocities u and v, in our solution, must be respectively even and odd in y. Accordingly ψ and ζ must be odd in respect of β,* and it will be sufficient to confine attention to the half-plane in which β is positive. The condition (27) at infinity will be satisfied if we make $\psi = \beta$, and this is a particular solution of (23), since it makes ζ vanish everywhere, by (24). Thus our complete solution will have the form

$$\psi = \beta + \psi_1, \qquad (30)$$

where ψ_1 is a solution of (23) and (24), *vanishing with* β, of which the differentials with respect to α and β must tend to zero at an infinite distance (in any direction) from the plate. The first of the conditions (26) at the boundary of the plate will be satisfied; the second imposes the relation

$$\frac{\partial \psi_1}{\partial \beta} = -1 \qquad (31)$$

at all points on the surface of the plate,—*i.e.*, lying within that part of the line ($\beta = 0$) for which $\alpha_1 > \alpha > \alpha_2$.

10. Making use of the second of (28), we may write equation (24) in the form

$$\left[\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right] \psi = \frac{\zeta}{U^2}. \qquad (32)$$

* ψ must in fact vanish with β, and ζ, since it must be continuous at all points in the fluid, will also vanish at points on the line ($\beta = 0$) which are outside the range ($\alpha_2 < \alpha < \alpha_1$); but within this range ζ will be discontinuous, having equal and opposite values on opposite sides of the plate.

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If (to fix ideas) we now give to ν the value $\frac{1}{2}$, so that (22)—from which the governing equation (23) was deduced—can be written in the form

$$\left[\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}+2\frac{\partial}{\partial\alpha}\right]\zeta=0, \quad \ldots \ldots \quad (33)$$

we have in place of (29) the expression

$$R\left(=\frac{UL}{\nu}\right)=2(\alpha_1-\alpha_2). \quad \ldots \ldots \quad (34)$$

Conversely, if in the assumed solution (30) ψ_1 satisfies the same boundary conditions as before, but is now taken to be governed by (32) and (33) in place of (23) and (24), the REYNOLDS' number of the motion will be given by (34),—*i.e.*, it is determined by the length, *on the* $\alpha-\beta$ *net*, of that part of the line ($\beta=0$) which represents our thin plate. When we have fixed this length, we have settled the REYNOLDS' number of the motion investigated; and although (for convenience) we have adopted a definite value for ν, it is known from the theory of dynamical similarity that our solution will apply to any values of U, L and ν which satisfy the relation (34). The occurrence of U^2 in (32) is a matter of no consequence, since the absolute magnitude of ψ_1 is determined solely by the boundary condition (31), and the absolute magnitude of ζ is not restricted by (33).

Solution of the Equation in ζ. *The "Vorticity Doublet."*

11. The procedure indicated in the last paragraph is in fact equivalent to a change of variables, from α, β to $\alpha/2\nu$, $\beta/2\nu$. It permits the introduction of a standard solution of equation (33), which solution is also the basis of LAMB's treatment of the OSEEN equation.* This may be written in the form

$$\zeta_1=e^{-\alpha}K_0(r), \quad \ldots \ldots \quad (35)$$

where r stands for $\sqrt{\alpha^2+\beta^2}$, and $K_0(r)$ is the BESSEL's function of imaginary argument, of the "second kind," which has been tabulated.† It represents the solution of (33) appropriate to a point source of vorticity situated at the point ($\alpha=0$, $\beta=0$), which point (so far as the solution (35) is concerned) may be anywhere on the α - β net.

We use this solution to deduce a second solution ζ_2, appropriate to a "doublet of vorticity." Let a point source of vorticity having a *positive* strength Σ be situated at ($\alpha=0$, $\beta=\varepsilon$), and a point source of vorticity having a *negative* strength $-\Sigma$ be

* *Loc. cit.*, § 343. It will be observed that equation (33), which is the governing equation in the particular problem now under discussion, has the form of OSEEN's equation (12).

† *Cf.*, WATSON's "*Theory of Bessel Functions*," Camb. (1923), or JAHNKE and EMDE, "*Funktionentafeln*," Leipzig (1909).

situated at ($\alpha = 0$, $\beta = -\varepsilon$). The solution of (33) appropriate to the positive source is

$$\zeta = \Sigma\, e^{-\alpha}\, K_0(r_1),$$

where r_1 stands for $\sqrt{\alpha^2 + (\beta - \varepsilon)^2}$; and the solution appropriate to the negative source is

$$\zeta = -\Sigma\, e^{-\alpha}\, K_0(r_2),$$

where r_2 stands for $\sqrt{\alpha^2 + (\beta + \varepsilon)^2}$: hence the combined solution, *when* ε *is made infinitesimally small but* $\varepsilon\Sigma$ *is kept finite*, is given by

$$\zeta_2 = \Sigma\, e^{-\alpha}\, \mathrm{Lt}_{\varepsilon=0}\left[2\varepsilon \frac{\partial}{\partial \varepsilon} K_0(r_1)\right], \text{ where } r_1 \text{ has the significance given above,}$$

$$= \Sigma\, e^{-\alpha}\, \mathrm{Lt}_{\varepsilon=0}\left[-2\varepsilon K_1(r_1)\frac{\partial r_1}{\partial \varepsilon}\right], \text{ by the property of the Bessel functions,}$$

$$= 2\Sigma\varepsilon\,\frac{\beta}{r} e^{-\alpha} K_1(r), \text{ where } r \text{ stands for } \sqrt{(\alpha^2 + \beta^2)} \text{ as before.}$$

Disregarding the constant factor, we may say that

$$\zeta_2 = e^{-\alpha}\,\frac{\beta}{r}\,K_1(r) \quad \ldots \ldots \ldots \ldots \quad (36)$$

is a solution of (33) appropriate to a unit doublet situated at ($\alpha = 0$, $\beta = 0$). So far as the solution is concerned, this point may be anywhere on the α–β net.

Let the point be situated on the line which contains the flat plate under investigation (*cf.* § 9). It is easily verified that ζ_2 behaves, near ($r = 0$), like $\beta/(\alpha^2 + \beta^2)$; hence on this line ($\beta = 0$), except at the point itself, ζ_2 vanishes as we required it to do. For large values of r it behaves as

$$\sqrt{\frac{2}{\pi}} \cdot \frac{\beta e^{-(r+\alpha)}}{r^{3/2}}. \quad \ldots \ldots \ldots \ldots \quad (37)$$

12. The nature of the solution ζ_2 is exhibited by the contours plotted in fig. 1; the doublet of vorticity is situated at O, and since the contours all pass through this point they have been stopped at the boundary of a rectangle surrounding O, in order to avoid confusion. Numerals attached to the bolder curves indicate the relative values of ζ_2 on those contours; the absolute values are immaterial to the purposes of this paper. The contours in the region of negative β can be obtained by reflection of the diagram with respect to the line ($\beta = 0$); on them, of course, ζ_2 will have negative values.

The scale adopted for α is shown in the diagram, and a like scale holds for β. Referring to the first of (28), we see that these scales for α and β may be identified with the scales for x and y, provided that U is assumed to have the value -1 (*i.e.*, when the general direction of the fluid stream is from right to left, in the direction of α decreasing). On this understanding, fig. 1 may be interpreted as the vorticity pattern which results (according to the approximate equation (22), which is identical in form with Oseen's equation) when fluid flows past an infinitesimally thin cylinder, *of any symmetrical cross-*

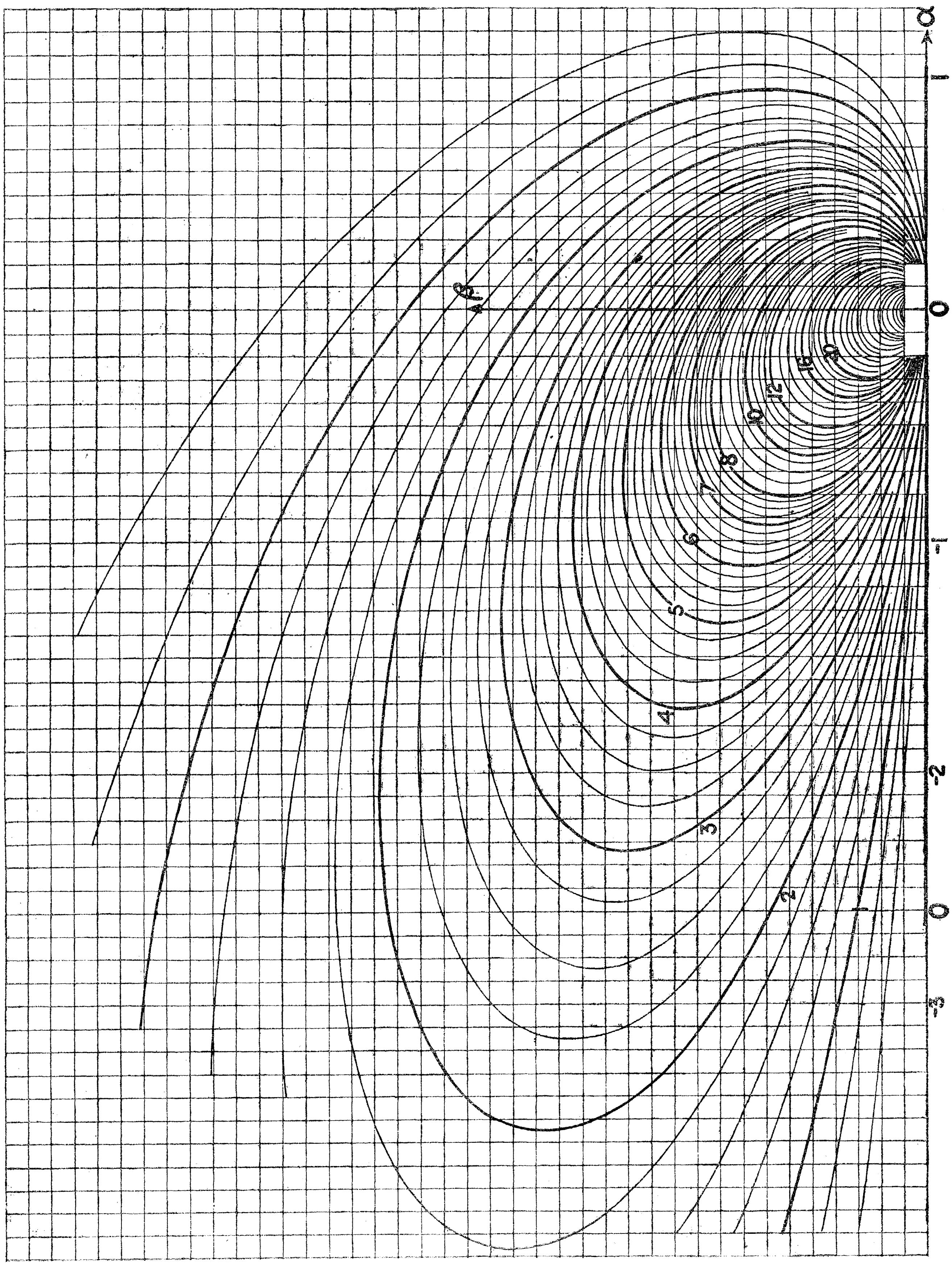

Fig. 1.

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section, which is situated at the origin; for it is evident that the viscous drag on the surface of such a cylinder must generate vorticity in a matter which can be represented by our "vorticity doublet."

An argument on the lines of § 10 may be employed to give this solution a wider application. The REYNOLDS' number of the motion (expressed in terms of the "up-stream length" of the cylinder) is of course infinitesimal; but, referring to (34), we see that the scale of the vorticity pattern is such that the length denoted by unity on the α-scale corresponds with a distance L in the fluid stream, where

$$\frac{\mathrm{UL}}{\nu} = 2. \qquad (38)$$

Thus the length denoted by unity in fig. 1 corresponds with a length $2\,\nu/\mathrm{U}$ in the motion represented, where ν is the kinematic viscosity of the fluid and U the velocity of the undisturbed stream.*

"Standard" Solutions for ζ.

13. It is not difficult, after inserting the solution ζ_2 in the right-hand side of (32), to integrate the resulting equation and so to obtain a solution satisfying all except (31) of the conditions imposed (in § 9) upon our required solution ψ_1;† from this we could proceed, by a method of trial and error, to construct an approximate solution of our problem, assuming the vorticity ζ to be suitably distributed along the surface of the plate. But, for reasons which will appear in Part III of this paper, we have adopted a different procedure in developing our approximate methods; and our next step is to construct diagrams, of the kind of fig. 1, to represent the vorticity patterns which result by convection—according to the approximate equation (33)—when ζ has certain specified distributions along the surface of the plate.

In each of these distributions ζ has equal and opposite values on opposite sides of the plate; accordingly they may be regarded as distributions of "vorticity doublets," varying in strength from point to point, along that part of the line $(\beta = 0)$ which represents the plate. When the variation of doublet strength along the plate is given, it is an easy matter to derive from fig. 1 (or from the calculations upon which this diagram is based) the corresponding value of ζ at any point in the fluid field; for if the doublet intensity has a value $\sigma(\alpha')$ at a point $(\alpha', 0)$ on that part of the line $(\beta = 0)$ which lies between $(\alpha = \pm\,\alpha_1)$, and is zero outside this range, it is evident that at any point (α, β) in the field ζ will be given by

$$\zeta = \int_{-\alpha_1}^{\alpha_1} \sigma(\alpha') \,.\, \mathrm{Z}\,\{(\alpha - \alpha'),\ \beta\}\, d\alpha', \qquad (39)$$

where $\mathrm{Z}(\alpha, \beta)$ stands for that function of α, β which has hitherto been denoted by ζ_2. The integration is easily performed by graphical or approximate numerical methods.

Figs. 2–6 exhibit distributions of ζ which have been determined in this way. They

* It will be remembered that ν has the dimensions L^2/T.

† We have in fact used this means to check the accuracy of our approximate methods. *Cf.* § 19 *infra*, and § 6 of the paper by BAIRSTOW, CAVE and LANG cited in § 9.

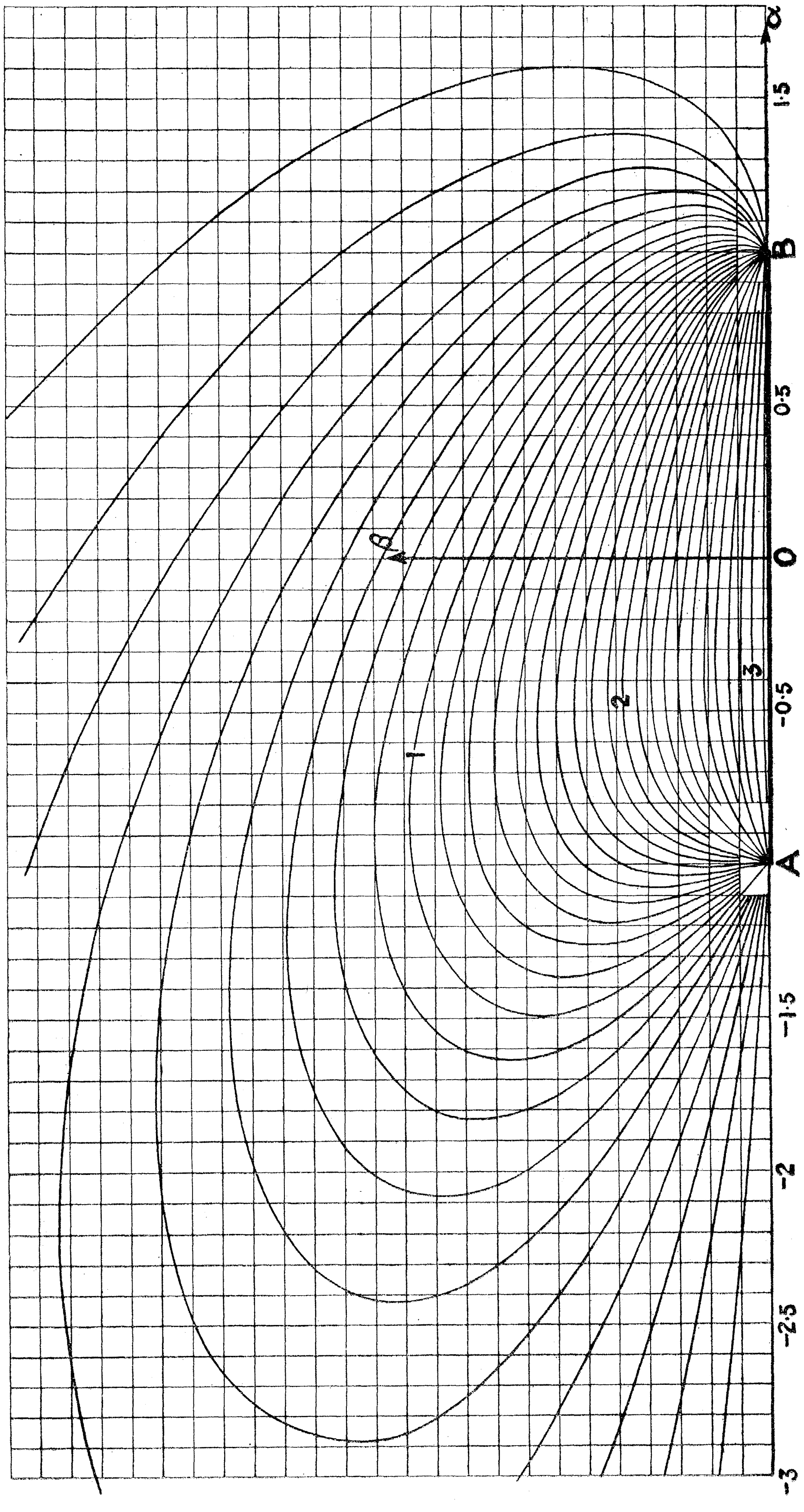

Fig. 2.

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refer respectively to constant, linear, quadratic, cubic and quartic distributions of doublet strength along that part (AB in the diagram) of the line ($\beta = 0$) for which

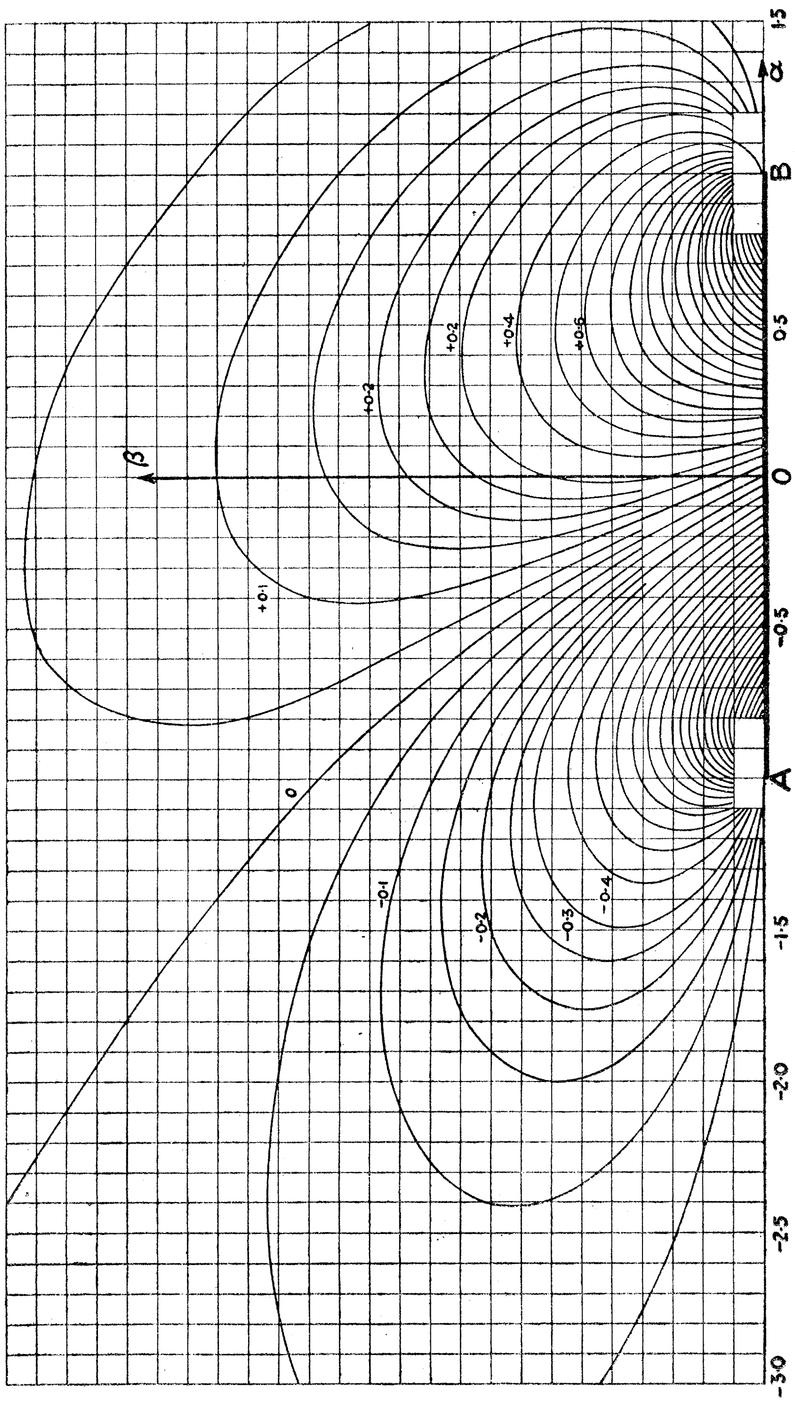

Fig. 3.

$-1 < \alpha < 1$. *Referring to* (34), *we observe that the corresponding value of* Reynolds' *number is* $R = 4$.

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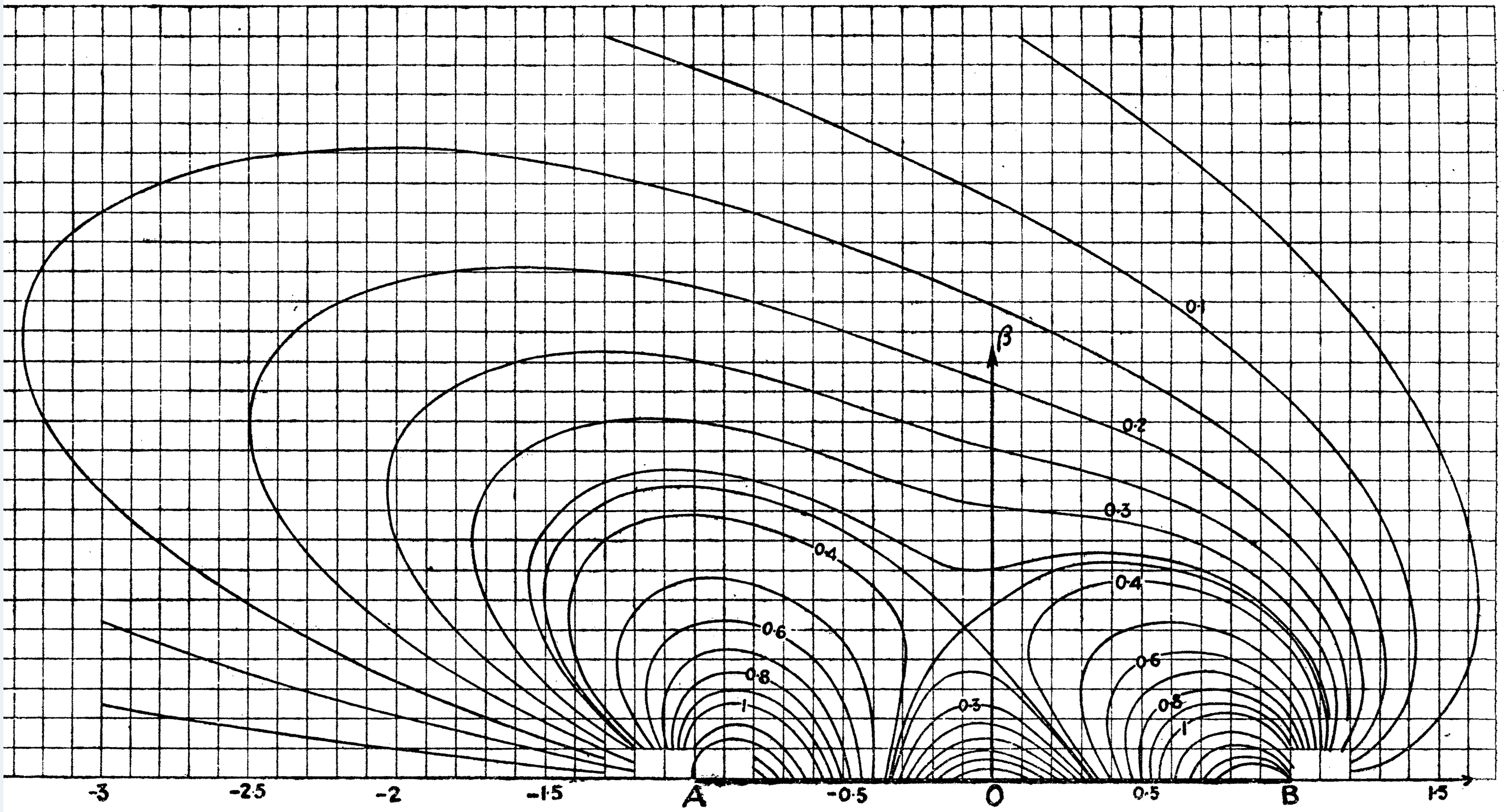
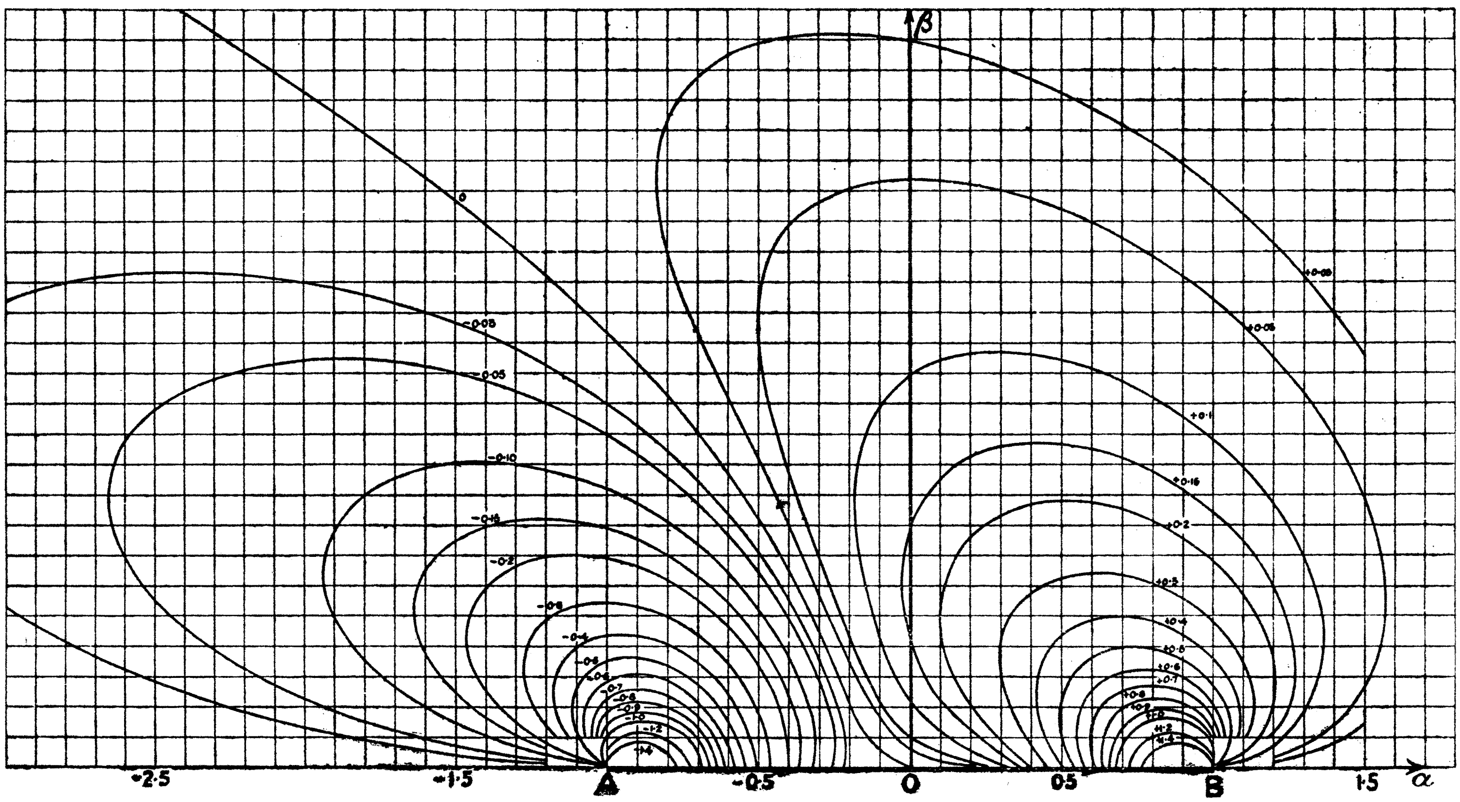


0·1
0·2
0·3
0·4
0·4
0·6
0·8
1
0·3
0·6
0·8
1
β
-3
-2·5
-2
-1·5
A
-0·5
O
0·5
B
1·5

FIG. 4.



β
-2·5
-1·5
A
-0·5
O
0·5
B
1·5
α

FIG. 5.

In fig. 2, ζ has a constant value π along AB. In figs. 3–6, ζ rises from a zero value at O to π (or $-\pi$) at A and B.

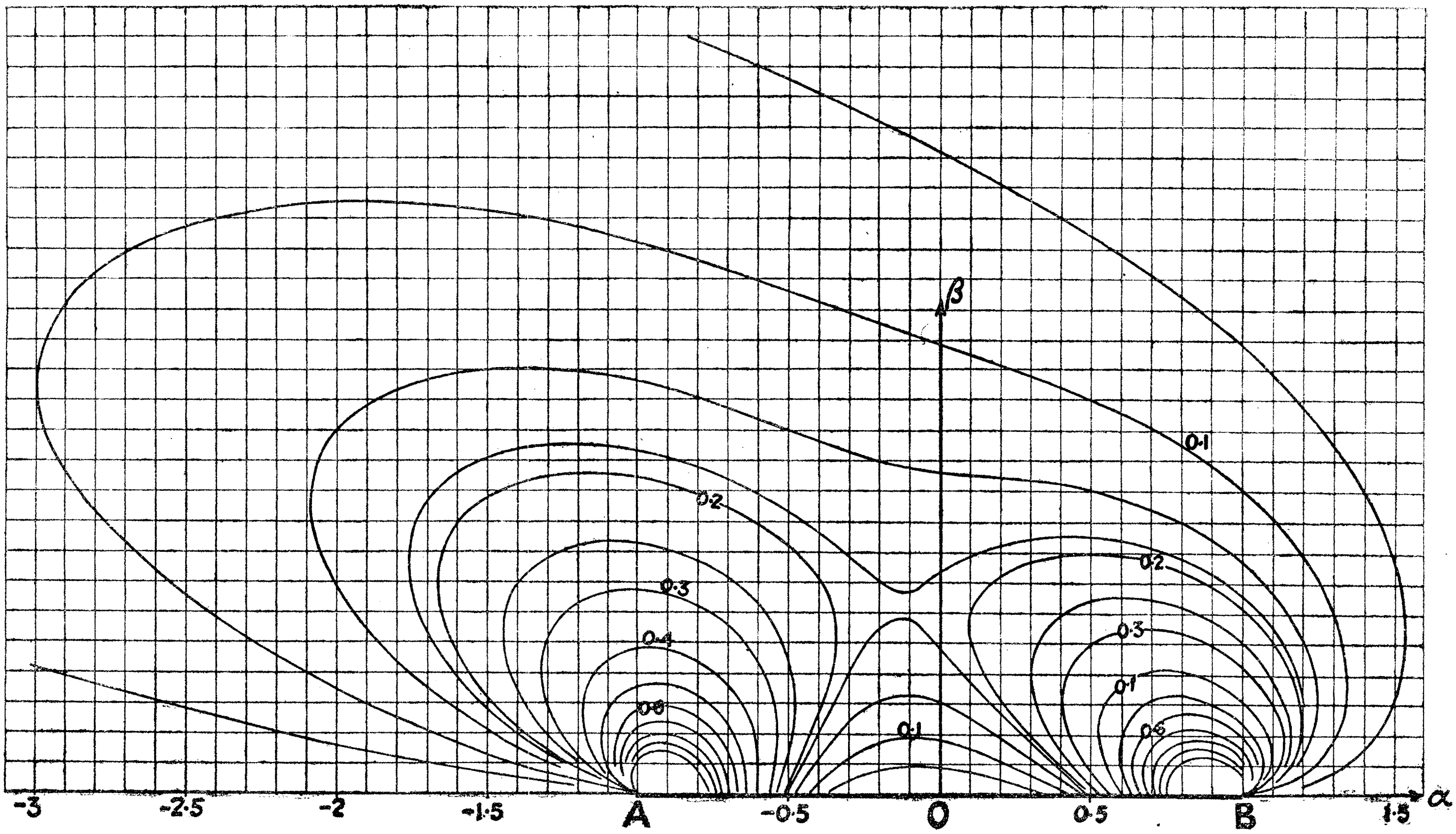


FIG. 6.

Derivation of the Stream Function ψ.

14. Corresponding with each of these five standard distributions of vorticity we can deduce, from (32), a solution for ψ which vanishes with β and which tends to zero at an infinite distance from the origin. Combining the five ψ-solutions in appropriate proportions, we shall be able to satisfy (31) at any five points in the range $(-1 < \alpha < 1)$, and so to arrive at an approximate solution for ψ_1. All that (for this purpose) we shall need to know about the standard ψ-solutions is the variation of $\left(\frac{\partial \psi}{\partial \beta}\right)_{\beta=0}$ throughout the range.

In any one of the distributions of vorticity, let $\zeta(\alpha', \beta')$ stand for the value of ζ at a point (α', β'), and let r be given by

$$r^2 = (\alpha - \alpha')^2 + (\beta - \beta')^2. \quad \ldots \ldots \ldots \ldots (40)$$

Then it is known from the theory of attractions that

$$U^2 \psi = \frac{1}{2\pi} \iint \zeta(\alpha', \beta') \log r \, d\alpha' \, d\beta' \quad \ldots \ldots \ldots (41)$$

(in which the double integral is to be taken over the whole of the fluid field) is a solution of (32) in which $\frac{\partial \psi}{\partial \alpha}$ and $\frac{\partial \psi}{\partial \beta}$ tend to zero at an infinite distance from the origin. When (as here) ζ is an odd function of β, ψ will vanish, by symmetry, at all points on the line ($\beta = 0$).

From (41) we deduce that

$$U^2 \frac{\partial \psi}{\partial \beta} = \frac{1}{2\pi} \iint \zeta\,(\alpha', \beta')\, \frac{\beta - \beta'}{r^2}\, d\alpha' d\beta', \quad \ldots \ldots \ldots \quad (42)$$

and on the line ($\beta = 0$) this equation becomes

$$- U^2 \left(\frac{\partial \psi}{\partial \beta}\right)_{\beta = 0} = \frac{1}{2\pi} \iint \zeta\,(\alpha', \beta')\, \frac{\beta'}{(\alpha - \alpha')^2 + \beta'^2}\, d\alpha' d\beta' \quad \ldots \quad (43)$$

The integral in (43) is to be taken over the whole field; but, since ζ is an odd function and $\zeta \,.\, \beta'$ accordingly an even function of β', we may actually perform the integration over the upper half-plane only, and double the result.

15. The integration may be effected by graphical methods, using contour diagrams of ζ such as have been given in figs. 2–6. Referring to fig. 7, and writing

$$\left.\begin{aligned} \alpha - \alpha' &= r \cos \theta, \\ \beta' &= r \sin \theta, \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (44)$$

we have from (43)

$$- U^2 \left(\frac{\partial \psi}{\partial \beta}\right)_{\beta=0} = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \frac{\zeta \beta'}{r^2}\, d\alpha'\, d\beta',$$

$$= \frac{1}{\pi} \int_0^\pi \sin \theta\, d\theta \int_0^\infty \zeta\, dr.$$

Now we have (integrating by parts)

$$\int_0^\infty \zeta\, dr = \Big[\zeta \,.\, r\Big]_0^\infty - \int r\, d\zeta,$$

and the term in square brackets vanishes at both limits. So we may write

$$U^2 \left(\frac{\partial \psi}{\partial \beta}\right)_{\beta=0} = \frac{1}{\pi} \int_0^\pi \sin\theta\, d\theta \int r\, d\zeta,$$

$$= \frac{1}{\pi} \int d\zeta \int_0^\pi r \sin\theta\, d\theta, \quad \ldots \ldots \ldots \quad (45)$$

if we change the order of integration,—as is permissible, since ζ tends exponentially to zero as r tends to infinity.

The integration $\int r \sin\theta\, d\theta$, in (45), is to be performed along a path on which ζ is constant—*i.e.*, along the contours of figs. 2–6. The integral is a function of ζ and of α (the abscissa of P in fig. 7): plotting it against ζ, we can perform the second integration by means of SIMPSON's rule or by "counting squares," and so deduce the value of $\partial \psi / \partial \beta$ for the selected point P.

Within the rectangle

$$\left.\begin{array}{r} -3 < \alpha < 1\cdot 5 \\ -2\cdot 5 < \beta < 2\cdot 5 \end{array}\right\} \quad \ldots \ldots \ldots \ldots (46)$$

the first integration (along the ζ-contour) was performed by means of a mechanical integrator, specially devised for this purpose, which employed the mechanism of an ordinary Amsler planimeter.* The contribution to (45) of the region outside the rectangle (*cf.* § 25) was determined by drawing contours of ζ_2 (fig. 1) to a reduced scale, integrating as above to find $\left(\frac{\partial\psi}{\partial\beta}\right)_{\beta=0}$, and *finally* extending the results (by the method of § 13) to take account of the distributed vorticity doublets.† As $\zeta \to 0$ the value of $\int r \sin\theta \, d\theta \to \infty$, and allowance for this must be made in the calculations. It will be shown in the sequel that no appreciable error is involved.

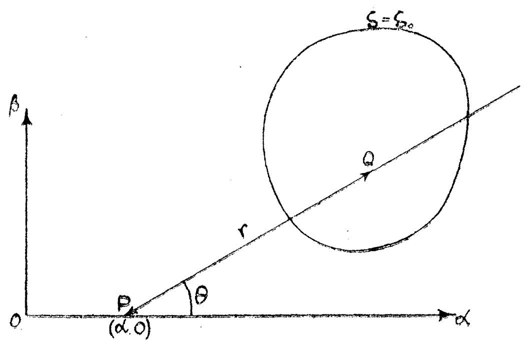


FIG. 7.

16. Before proceeding to the approximate satisfaction of (31), it will be well to consider the dimensional aspects of (45). We have seen (§ 13) that our calculations relate to a definite value of REYNOLDS' number (R = 4), but (§ 10) that within this restriction they have the range predicted by dimensional theory; accordingly it is legitimate, in order to fix ideas, to assign any convenient values to U, L and ν severally which give the required value for REYNOLDS' number. As in previous paragraphs, we shall assume the values

$$U = -1, \quad \nu = \tfrac{1}{2}, \quad \ldots \ldots \ldots \ldots (47)$$

so that the scales of α and β can also be interpreted as scales for x and y. Then we see (from figs. 2–6) that L has the value 2, so that everything is consistent.

On this understanding, we have from (8) and (28)

$$u = -\frac{\partial\psi}{\partial y} = U\frac{\partial\psi}{\partial\beta},$$

$$= -\frac{\partial\psi}{\partial\beta}, \quad \ldots \ldots \ldots \ldots (48)$$

* The instrument is described in the Appendix, p. 63.

† *Cf.* § 25.

and equation (31) requires that

$$u_1 = 1, \text{ at all points in the range } (-1 < \alpha < 1), \quad \ldots \ldots \quad (49)$$

where u_1 is the velocity corresponding with our required solution ψ_1, according to equation (48). This is the condition which we have to satisfy approximately.

We have also, from (48) and (45)—in which U^2 is to be given the value unity—

$$u(\alpha, 0) = -\left(\frac{\partial \psi}{\partial \beta}\right)_{\beta=0} = -\frac{1}{\pi}\int d\zeta \int r \sin\theta \, d\theta. \quad \ldots \ldots \quad (50)$$

Approximate Solution for the Flat Plate.

17. Now let $(u)_n$ stand for the value of $u(\alpha, 0)$ which is given by the approximate integration when the doublet strength σ (§ 13) is assumed to vary as α^n in the range $(-1 < \alpha < 1)$. We have calculated values of $(u)_0$, $(u)_1$, $(u)_2$, $(u)_3$, $(u)_4$ at five different points in this range, namely, the points

$$\alpha = -1, \; -0\cdot5, \; 0, \; 0\cdot5, \; 1. \quad \ldots \ldots \quad (51)$$

Hence, assuming as an approximate form for u_1

$$u_1 = a_0 (u)_0 + a_1 (u)_1 + a_2 (u)_2 + a_3 (u)_3 + a_4 (u)_4, \quad \ldots \ldots \quad (52)$$

we can write down five equations, representing the conditions for the satisfaction of (49) at the same five points. Solving these equations, we can deduce values for $a_0, a_1, \ldots a_4$.

Our calculations gave for the five equations :—

$$\left.\begin{aligned}
1 &= 2\cdot039a_0 - 0\cdot362a_1 + 0\cdot776a_2 - 0\cdot245a_3 + 0\cdot500a_4,\\
1 &= 2\cdot331a_0 - 0\cdot251a_1 + 0\cdot691a_2 - 0\cdot079a_3 + 0\cdot374a_4,\\
1 &= 2\cdot217a_0 + 0\cdot184a_1 + 0\cdot543a_2 + 0\cdot098a_3 + 0\cdot305a_4,\\
1 &= 1\cdot915a_0 + 0\cdot531a_1 + 0\cdot614a_2 + 0\cdot251a_3 + 0\cdot343a_4,\\
1 &= 1\cdot285a_0 + 0\cdot458a_1 + 0\cdot556a_2 + 0\cdot307a_3 + 0\cdot382a_4.
\end{aligned}\right\} \quad \ldots \ldots \quad (53)$$

They are satisfied by the values

$$\left.\begin{aligned}
a_0 &= 0\cdot3158,\\
a_1 &= -0\cdot1853,\\
a_2 &= -0\cdot829,\\
a_3 &= 0\cdot863,\\
a_4 &= 2\cdot290.
\end{aligned}\right\} \quad \ldots \ldots \quad (54)$$

Accuracy of the Approximate Solution.

18. This completes our approximate solution, and it remains to discuss the results. As was mentioned in §9, the problem of the flat plate (for which our form of the governing equation is identical with OSEEN'S; *cf.* footnote, §11) has been investigated analytically by BAIRSTOW and others; accordingly we can use their solution to test the accuracy of the approximate methods which have been developed in §§13–17, before we proceed to apply these in cases which have not hitherto been treated on the basis of our governing equation (16).

Combining in proportions fixed by (54) the distributions of vorticity which have been given in figs. 2–6, we could deduce the actual distribution which our solution implies, and from this proceed to determine the corresponding stream function. The labour involved would be considerable, and very little additional information would result. What is more important is to examine the conditions which our solution implies at the surface of the plate.

Our boundary condition (49) requires that u_1 shall have the value unity at all points on the plate; if in fact the solution makes u_1 equal to $(1 + \varepsilon)$ at any point (ε being a quantity which varies with α), it is inaccurate to the extent that at this point it implies a velocity (of the fluid relative to the plate) which is given by $-\varepsilon U$, and accordingly is up-stream when ε is positive.* Fig. 8 shows the variation of u_1, as

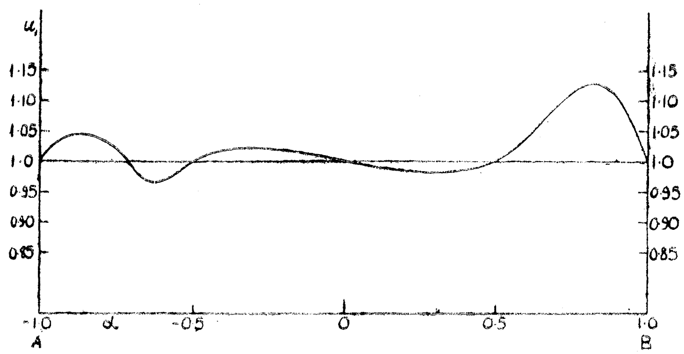


FIG. 8.

obtained in our approximate solution, along the plate AB; it will be seen that the errors are satisfactorily small excepting near the leading edge, where they amount to about 13 per cent. This result suggests that it might have been better, having committed ourselves to a polynomial expression for the "doublet strength" $\sigma(\alpha)$, not to have chosen equidistant points, as in (51), at which to satisfy (49); but calculations in which (51) was replaced by

$$\alpha = -1, -0\cdot5, 0, 0\cdot7, 1$$

* We have from (48) and (30),

$$u = U\frac{\partial\psi}{\partial\beta} = U\left(1 + \frac{\partial\psi_1}{\partial\beta}\right) = U(1 - u_1).$$

gave no closer approximation. In the case of the circular cylinder (to be discussed in Part III) it will be found that the polynomial expression has given more satisfactory results.

Fig. 9 shows the calculated intensity of the doublet strength along the plate AB when, as in preceding paragraphs, U is assumed to have the value —1, so that the flow of the fluid is from right to left. The actual intensity will of course be proportional to U.

The full-line curve in fig. 10 has been drawn for comparison with the curve marked ($k = 1$) in fig. 6 of the paper by BAIRSTOW and others (their curve, which relates to the

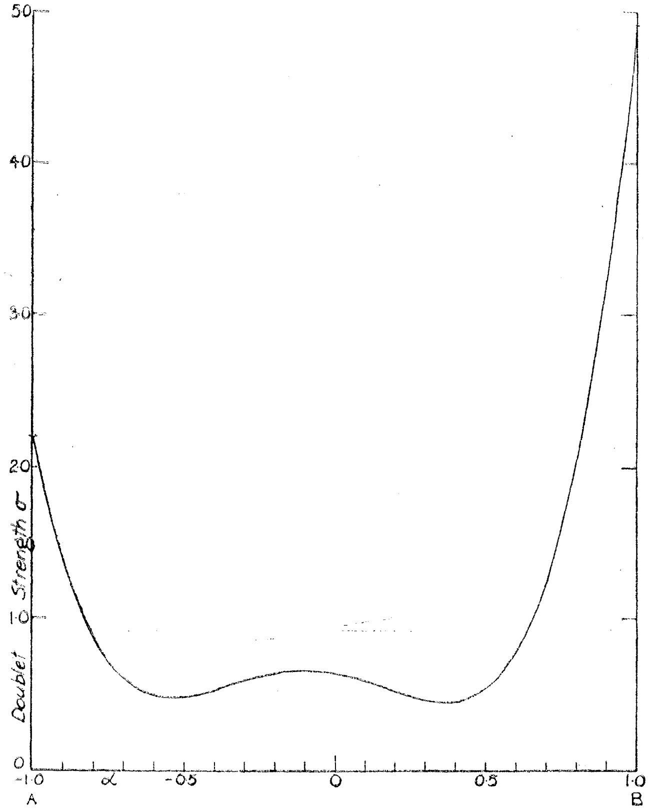


FIG. 9.

same value ($R = 4$) of REYNOLDS' number, has been indicated by dotted lines). In effect it shows the variation along the plate AB of $\int_{\alpha}^{1} \zeta(\alpha')\,d\alpha'$, the integral of the vorticity (a quantity denoted by $-\chi_{E}/U$ in the paper cited).*

In the paper cited it was concluded (pp. 410, 411) that the vorticity (*i.e.*, the gradient of the curve in fig. 10) attains infinite values at the ends of the plate. The assumption of a polynomial form for $\sigma(\alpha)$ excludes the possibility of realizing this conclusion in our solution, and it may be that our relatively large errors in the neighbourhood of the leading edge are attributable to this fact. We are not, however, convinced that the conclusion is correct,—both on physical grounds and because the arguments for it

* On the plate $\zeta(\alpha') = \pi\sigma(\alpha')$.

given in the paper (which are based on analytical considerations) seem to us to be open to criticism. We think it more probable that the vorticity, although it varies rapidly near the ends of the plate, is finite at all points in the field.

19. The foregoing comparison has not been entirely satisfactory, and before applying our approximate methods to other problems (for which our governing equation is new, and hence no comparable solutions exist) it seems desirable to examine their probable accuracy from a theoretical standpoint. We are not of course concerned, in this paragraph, with the accuracy of (33) regarded as an approximation to the exact equation of motion, but only with the methods of solution employed in §§13–17.

Three possible sources of error occur in these methods:—(1) the distributions of vorticity given in figs. 2–6 have been deduced by numerical integration from the basic (and exact) solution ζ_2; (2) the integration of (45), to obtain the quantities denoted

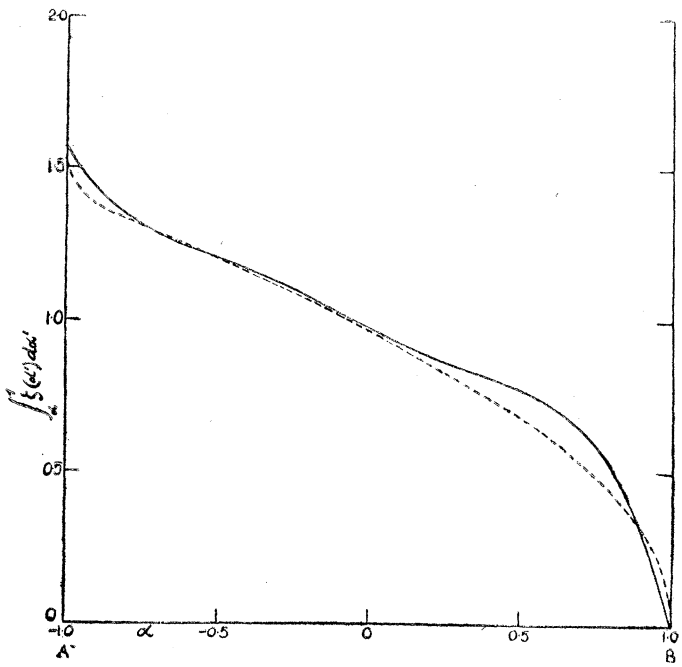


FIG. 10.

by $(u)_0$, $(u)_1$, . . . etc., has been performed graphically (in the region close to the plate); (3) the boundary condition (49) has been satisfied at five points only in the length of the plate.

Errors arising from cause (1) may be expected to be very small; they will not be systematic, and accordingly should be smoothed out when the diagrams of vorticity are drawn.

Errors arising from cause (2) can be examined in the light of a comparison with exact analytical solutions. For a unit doublet situated at the origin, u_1 is given by*

$$u_1 = 2e^{-\alpha} \, \mathrm{K}_0(r) + \frac{\partial}{\partial\alpha}\left[e^{-\alpha} \, \mathrm{K}_0(r) + \log r\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

* BAIRSTOW etc., *loc. cit.*, p. 407, equation E (5).

and it follows that a distribution of doublet strength $\sigma(\alpha')$ extending over the range $(-1 < \alpha' < 1)$ will lead to

$$u_1 = \int_{-1}^{1} \sigma(\alpha') d\alpha' \left\{ 2e^{-(\alpha-\alpha')} \mathrm{K}_0(r') + \frac{\partial}{\partial \alpha} \left[e^{-(\alpha-\alpha')} \mathrm{K}_0(r') + \log r' \right] \right\}, \quad . \quad . \quad (56)$$

where the symbols have the meanings given previously. When $\sigma(\alpha')$ has the form α'^n, the definite integral can be quickly evaluated, by SIMPSON's rule, after values of the integrand have been calculated for a number of points in the range.* The result is a set of values for $(u)_0$, $(u)_1$, . . . etc. which can be compared with the set obtained previously. A comparison is made in Table I, where values found by our approximate methods are given above values determined in the manner here described.

TABLE I.

α	$(u)_0$	$(u)_1$	$(u)_2$	$(u)_3$	$(u)_4$
− 1·0	2·039 2·039	− 0·362 − 0·362	+ 0·776 + 0·777	− 0·245 − 0·248	+ 0·500 + 0·501
− 0·5	2·331 2·330	− 0·251 − 0·250	+ 0·691 + 0·688	− 0·079 − 0·081	+ 0·374 + 0·373
0	2·217 2·217	+ 0·184 + 0·184	+ 0·543 + 0·540	+ 0·098 + 0·100	+ 0·305 + 0·302
+ 0·5	1·915 1·913	+ 0·531 + 0·535	+ 0·614 + 0·616	+ 0·251 + 0·244	+ 0·343 + 0·345
+ 1·0	1·285 1·289	+ 0·458 + 0·454	+ 0·556 + 0·557	+ 0·307 + 0·311	+ 0·382 + 0·381

The agreement is good, particularly as regards $(u)_0$; and experience of the approximate methods indicates that the maximum errors here shown for the higher terms (2 per cent., except in one case) have arisen from the rapid variation of ζ near the ends of the plate. The differences are so small and unsystematic that it seems safe to conclude that errors from cause (2) are unimportant.

Errors of type (3) are an unavoidable feature of our method of solution: all that can be said is that (on physical grounds) they may be expected to decrease rapidly as the number of points is increased at which (49) is satisfied. The order of accuracy attainable with five points has been discussed in § 18.

* Care is necessary at the point $\alpha = \alpha'$, where the integrand is logarithmically infinite.

PART III.—SOLUTION OF THE APPROXIMATE EQUATION FOR THE CASE OF FLOW PAST A CIRCULAR CYLINDER.

Applicability of the Standard Solutions in ζ.

20. Reverting to § 8, we observe that in that paragraph no restriction is imposed upon the shape of the solid cylinder, except that this must be symmetrical with respect to the direction of the undisturbed flow; with this one proviso, if α and β are taken to be the velocity potential and stream function for irrotational flow past the cylinder in question, then *in every case* ζ is governed by (22) or by the equivalent equation (23), ψ is related with ζ by (24), and the boundary conditions are expressed by (26) and (27). According to OSEEN's approximate equation (12), ζ and ψ are governed by equations which have the same form for all shapes of cylinder when expressed in Cartesian co-ordinates: in our modification, the equation in ζ has a standard form when expressed in terms of α and β, but these functions must be given forms appropriate to the shape of cylinder in question.

Now we have already (in the distributions of vorticity which are represented by figs. 2–6) forms for ζ, regarded as a function of α and β, which satisfy equation (33); and this equation is the form assumed by (22) when ν is given the value $\frac{1}{2}$. If we transform the rectilinear α–β net of figs. 2–6 into the curvilinear α–β net appropriate to the shape of cylinder considered, the contours of ζ when correspondingly transformed (*i.e.*, drawn to pass through corresponding points on the new net) will represent new distributions of vorticity which (for $\nu = \frac{1}{2}$) are solutions of our governing equation (14) appropriate to the problem under discussion. In each of the new distributions ζ will be an odd function of β; it will vanish on that part of the line ($\beta = 0$) which represents the line of symmetry of the flow; and it will be finite on that part of the line ($\beta = 0$) which represents the boundary of the cylinder, taking equal and opposite values at points which are opposite to one another in relation to the line of symmetry. The stagnation points at front and rear of the cylinder will have the same co-ordinates ($\alpha = \pm\,\alpha_1$, $\beta = 0$) as the points A, B in the rectilinear net.

21. Suppose now that α, β, *in the curvilinear net*, are related with the Cartesian co-ordinates x, y by the equation

$$\alpha + i\beta = f(z), \qquad (57)$$

where $z = x + iy$. Knowing the form of $f(z)$, we can deduce the "velocity at infinity," and (since $\alpha = f(x)$ when $\beta = y = 0$) we can relate the up-stream dimensions $(x_1 - x_2)$ of the cylinder with the length $(2\alpha_1)$ of AB in the rectilinear net. Hence, since ν has already been given the value $\frac{1}{2}$, we know the value of REYNOLDS' number for the flow to which our ζ-distributions will apply.*

* *Cf.* the argument of § 10.

Thus in the case of the *circular cylinder* (with which the remainder of this section of the paper will be concerned) equation (57) takes the form (LAMB, *loc. cit.*, § 68)

$$\alpha + i\beta = \mathrm{U}\left(z + \frac{a^2}{z}\right), \quad \ldots \ldots \ldots \ldots (58)$$

where U is the velocity at infinity; therefore at the stagnation points ($x = \pm a$, $y = 0$) α will have the values $\pm 2\mathrm{U}a$. Now at the points A, B, in figs. 2–6, α has the values ± 1: accordingly those diagrams will apply to a circular cylinder of diameter d, provided that $\mathrm{U}d = 1$; that is to say, they will apply when the REYNOLDS' number of the flow past its circular cylinder is given by

$$\mathrm{R} = \frac{\mathrm{U}d}{\nu} = 2. \quad \ldots \ldots \ldots \ldots (59)$$

Formulation of the Problem. Outline of the Approximate Solution.

22. The stream-function ψ for the circular cylinder is related with ζ by equation (24), in which h^2 is given by (20). It is subject to the boundary conditions (26) and (27).

As in the problem of the flat plate (§ 9), we seek a solution for ψ of the form

$$\psi = \beta + \psi_1, \quad \ldots \ldots \ldots \ldots (30)\ bis$$

where ψ_1 is a solution of (24), *vanishing with* β, of which the differentials with respect to α and β tend to zero when $(\alpha^2 + \beta^2) \to \infty$. The condition (27) and the first of (26) will then be satisfied; the second of (26) will impose the relation

$$\frac{\partial \psi_1}{\partial \beta} = -1 \quad \ldots \ldots \ldots \ldots (31)\ bis$$

at points given by

$$-1 < \alpha < 1, \ \beta = 0; \quad \ldots \ldots \ldots \ldots (60)$$

and this remaining condition we shall satisfy *approximately* (*i.e.*, at five points in the range), by a combination of particular solutions each of which satisfy all the other conditions.

These particular solutions we derive from (24) by inserting for ζ, in that equation, the particular forms which are represented by figs. 2–6. The integrations are performed *in relation to the rectilinear* α–β *net*, by approximate (graphical) methods which have been described already (§§ 14–15).

23. It is necessary for this purpose to have contour diagrams for the quantity ζ/h^2, drawn on the rectilinear net. In general, whilst it is an easy matter to express h^2 in terms of x and y,* the derivation of an explicit form involving α and β only is very difficult. But this is not really essential; for we can use the Cartesian expression to

* Thus the Cartesian expression for h^2, in the case of the circular cylinder to which (58) refers, is easily shown to be

$$h^2 = \mathrm{U}^2\left\{\left(1 + \frac{a^2}{r^2}\right)^2 - 4\,\frac{a^2x^2}{r^4}\right\},$$

plot contours of constant h^2 on the curvilinear $\alpha-\beta$ net, and then we can easily transfer these contours (on the drawing board) to the rectilinear net; finally, having the contours both of ζ and of h^2, we can construct contours of the quantity ζ/h^2.

Fig. 11 will serve to illustrate the foregoing paragraph. Below, we have a contour of constant ζ drawn on the rectilinear $\alpha-\beta$ net; above, a curvilinear $\alpha-\beta$

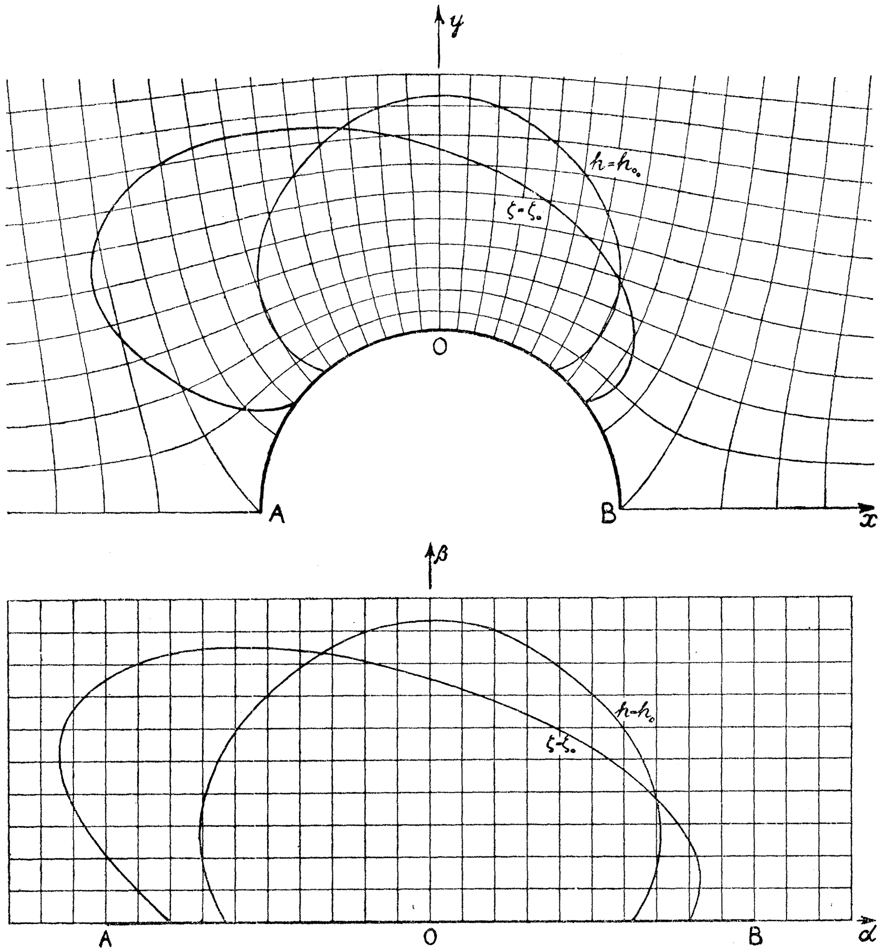


Fig. 11.

net is drawn in relation to a circular cylinder, and the ζ contour has been transferred to this. A line of constant h^2 is also shown in the upper diagram, and this is shown transferred to the rectilinear net below. Given a sufficient number of ζ and h^2 contours, the construction on this net of contours of constant ζ/h^2 is a problem which presents no difficulty.

Practical Details.

24. Actually the contours of constant ζ/h^2 were constructed by a method which did not involve the drawing of ζ and h^2 contours separately. Values of ζ had been calculated (for the construction of figs. 2–6) at a large number of points; accordingly the simplest

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procedure was to calculate values of h^2, and hence of ζ/h^2, for the same points, and from the tabulated values of ζ/h^2 to deduce contours by cross-plotting.

The doublet distributions shown in figs. 2–6 make ζ finite (and equal to $\pm\pi$; *cf.* § 13) at A and B, whereas in steady flow past a cylinder ζ must vanish at the stagnation points. This consideration (which was not appreciated when the solution was begun) suggested a slight change in procedure; instead of using doublet distributions for which

$$\sigma(\alpha) = \alpha^n, \qquad (61)$$

the type was taken to be of the form

$$\sigma(\alpha) = (1 - \alpha^2)\alpha^n, \qquad (62)$$

and new distributions of this type were derived from the old (figs. 2–6) by subtraction.

The number of component solutions available for building up a resultant distribution (in the manner of §17) was thus reduced to three;* but the accuracy of the final solution for the cylinder need not be expected, on this account, to be worse than that for the plate, because the double boundary condition (3) is now automatically satisfied at the stagnation points. Three arbitrary constants are available, and can be used to satisfy (49) at three intermediate points on the boundary.

It was decided to satisfy this second condition at points equidistant on the circular boundary,—that is, at eight equidistant points in all. Since on the boundary $\alpha = \cos\theta$ where $\tan\theta = y/x$, the values of α for the three intermediate points were $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$. The calculations required for the determination of u_1 follow the same lines as before (§§ 14–17), except that ζ is here replaced by ζ/h^2.

25. A reason now appears for the separate treatment (in § 15) of a finite rectangle embracing the points A, B in the rectilinear net. Outside this rectangle ζ is very small and h^2 very nearly constant; accordingly the exterior region contributes to u_1 an amount which is practically independent (for a given ζ-distribution) of the shape of the cylinder, and it is only in respect of the finite rectangle that new calculations are required.

Approximate Solution for the Circular Cylinder.

26. Let $(u)_n$ now stand for the value of $u(\alpha, 0)$ which results from the integration when the doublet strength varies between A and B according to equation (62). Values of $(u)_0$, $(u)_1$, $(u)_2$ were calculated for the points

$$\alpha = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, \qquad (63)$$

* Namely, those in which $\sigma = 1 - \alpha^2$, $\alpha(1 - \alpha^2)$, $\alpha^2(1 - \alpha^2)$. All five of the earlier distributions (figs. 2–6) are here involved.

and assuming as the approximate form for u_1

$$u_1 = b_0 (u)_0 + b_1 (u_1) + b_2 (u_2), \quad \ldots \ldots \ldots \quad (64)$$

we obtained the equations

$$\left.\begin{aligned} 1 &= 1\cdot231b_0 - 0\cdot1365b_1 + 0\cdot2728b_2, \\ 1 &= 1\cdot058b_0 + 0\cdot0499b_1 + 0\cdot1805b_2, \\ 1 &= 0\cdot819b_0 + 0\cdot1797b_1 + 0\cdot2064b_2. \end{aligned}\right\} \quad \ldots \ldots \quad (65)$$

These are satisfied by the values

$$\left.\begin{aligned} b_0 &= 0\cdot728, \\ b_1 &= 1\cdot148, \\ b_2 &= 0\cdot956, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (66)$$

and thus our solution for the circular cylinder is complete.

Accuracy of the Approximate Solution.

27. No previous results are here available as a check on the accuracy of our solution, and it does not appear practicable to formulate analytical expressions corresponding with (55) and (56). But since the same methods have been employed in this case as were previously applied to the flat plate, it seems reasonable to presume that the same order of accuracy has been attained.

Fig. 12, corresponding with fig. 9 for the plate, shows the distribution of doublet

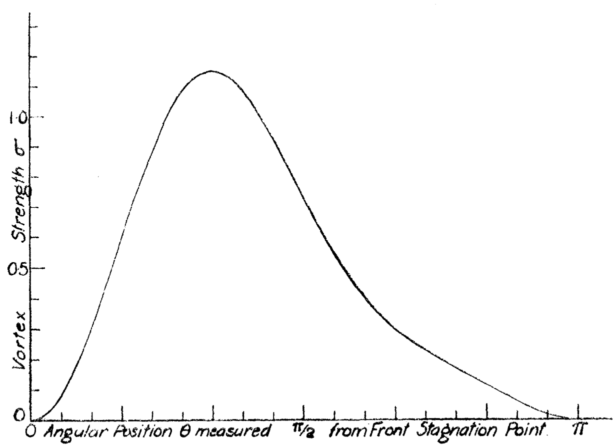
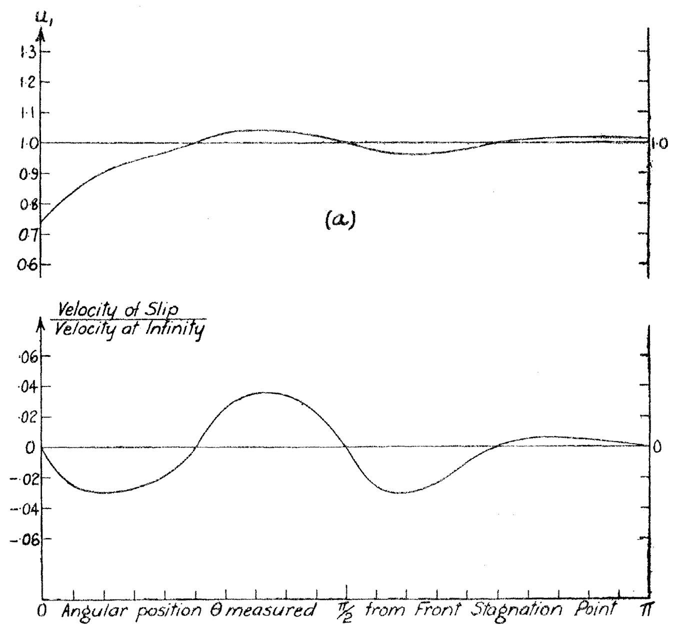


Fig. 12.

strength round the cylinder which our approximate solution implies. Fig. 13*a*, corresponding with fig. 8, exhibits the variation of u_1 (which should be unity everywhere) round the circumference.

If, as in § 18, we write $u_1 = 1 + \varepsilon$, the velocity of slip (*i.e.*, of the fluid in relation to the plate) is in this instance given by $-\varepsilon(u)_I$, where $(u)_I$ stands for the velocity at the point in question when the fluid is inviscid and the flow irrotational; $(u)_I$ is known, and accordingly the velocity of slip can be related with U, the velocity at infinity. Fig. 13*b* shows the velocity of slip expressed as a fraction of U, and plotted as a function of position on the circular boundary. It will be seen that the maximum deviation from unity is of the order of $3\frac{1}{2}$ per cent.; that is to say, the velocity of slip at the surface has not been made zero everywhere, but it has been reduced to about $3\frac{1}{2}$ per cent. of the velocity at infinity.



(*b*)

Fig. 13.

In this, as in the previous problem, closer approximation may be expected to result if more type solutions are employed, so that (49) can be satisfied at more points on the boundary. Considering that only three type solutions are here introduced, the results of this first investigation may be accounted satisfactory. It is evident that the polynomial form assumed for σ has given better results for the cylinder because the vorticity has not in this instance the rapid variations which characterize it near the ends of the flat plate. It may be anticipated that a like result will be obtained with any form having a fairly small curvature in the region of the forward stagnation point.

PART IV. GENERAL CONCLUSIONS.*

The Formula for Resistance.

28. From a physical standpoint the most important deduction which can be drawn from solutions such as we have obtained in this paper is the magnitude of the resultant force on the cylinder, per unit length. Referring to fig. 14, in which Ox is directed up-

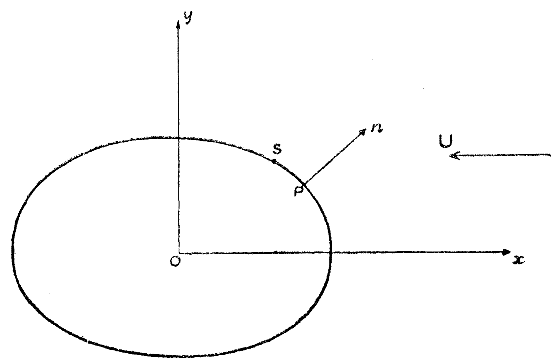


FIG. 14.

stream along the plane of symmetry of the cylinder, let Pn, the normal to its boundary at a point P, have direction cosines l and m in relation to Ox, Oy respectively, and let ds denote an element of the boundary. Then the force in question is given by †

$$\begin{aligned} D &= \int p_{nx}\, ds, \\ &= \int (l p_{xx} + m p_{xy})\, ds, \\ &= \int \left(-p + 2\mu \frac{\partial u}{\partial x}\right) dy - \mu \int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx, \qquad (67) \end{aligned}$$

the integration extending in both instances over the whole of the boundary. Making use of (1) and (5), and rejecting terms which make no contribution to the integrals, we can reduce this expression to the simpler form

$$D = -\int p dy + \mu \int \zeta\, dx, \qquad (68)$$

in which p, as in equations (2), denotes the "mean pressure." Here, as in (67), dx and dy are to be taken along the boundary.

* [*Added January* 27, 1933.—Paragraphs 29–32 of this section have been largely rewritten, following the discovery of a slip in our original work, which had led to an erroneous form for the expression (75). We are indebted to SIR HORACE LAMB for drawing our attention to this mistake.]

† *Cf.* LAMB, *loc. cit.*, §§ 325–326, which explain the notation used here.

Again, if we replace pdy, in the first of these integrals, by $d(py) - ydp$, the first term makes no contribution to the integral, and we are left with

$$\begin{aligned} \mathrm{D} &= \int y\,dp + \mu \int \zeta\, dx, \\ &= \int y \frac{\partial p}{\partial s} ds + \mu \int \zeta \frac{\partial x}{\partial s} ds. \qquad (69) \end{aligned}$$

29. In order to evaluate D we must have an expression for $\partial p/\partial s$, and accordingly it is necessary to consider what modification of the exact equations (2) is necessitated (for consistency) by our modification of the exact vorticity equation (11).

The body-forces X, Y being zero in our problem, it can be verified that these equations may be written in the forms*

$$\left.\begin{aligned} -v\zeta &= \nu\nabla^2 u - \frac{\partial \chi}{\partial x}, \\ u\zeta &= \nu\nabla^2 v - \frac{\partial \chi}{\partial y}, \end{aligned}\right\} \qquad (70)$$

where

$$\chi = \frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2). \qquad (71)$$

If in (70) we substitute for u and v from (13) *on the left-hand side only*, these equations become

$$\left.\begin{aligned} \frac{\partial \alpha}{\partial y} \cdot \zeta &= \nu\nabla^2 u - \frac{\partial \chi}{\partial x}, \\ -\frac{\partial \alpha}{\partial x} \cdot \zeta &= \nu\nabla^2 v - \frac{\partial \chi}{\partial y}, \end{aligned}\right\} \qquad (72)$$

Since α is plane harmonic, elimination of χ from these equations by cross-differentiation leads to our equation (14), and accordingly the relations (72) are consistent with that equation. Also by changing the co-ordinates in (72) from x, y, to α, β, we obtain

$$\left.\begin{aligned} -\nu \frac{\partial \zeta}{\partial \beta} &= \frac{\partial \chi}{\partial \alpha}, \\ \zeta + \nu \frac{\partial \zeta}{\partial \alpha} &= \frac{\partial \chi}{\partial \beta}, \end{aligned}\right\} \qquad (73)$$

whence it may be verified that χ, as found from (72), will satisfy an equation identical in form with our vorticity equation (22).

Now at the boundary, where u and v are zero, we have from (71)

$$p/\rho = \chi, \qquad (74)$$

* *Cf.* LAMB, *loc. cit.*, § 328, equations (6).

and it follows (since the boundary is a contour of constant β) that we may write

$$\int y \frac{\partial p}{\partial s} ds = \int y \frac{\partial p}{\partial \alpha} d\alpha,$$

$$= - \mu \int y \frac{\partial \zeta}{\partial \beta} d\alpha, \text{ by (74) and the first of (73).}$$

Accordingly we may write (69) in the form

$$D = \mu \left(- \int y \frac{\partial \zeta}{\partial \beta} d\alpha + \int \zeta \, dx \right), \quad \ldots \ldots \ldots \quad (75)$$

in which the integrations cover the whole of the cylindrical boundary.

30. It may be remarked that the same form will be obtained if we work on the basis of the exact equations (2) or (70). For provided that no body forces are operative, so that equations (70) still hold, the first of (73) will still hold *at the boundary* (where $u = v = 0$); hence, making use of (74), we have as before

$$\int y \frac{\partial p}{\partial s} ds = - \mu \int y \frac{\partial \zeta}{\partial \beta} \partial \alpha.$$

Resistance of the Flat Plate.

31. In relation to the flat plate (for which y vanishes at the boundary), some uncertainty exists regarding the expression which should be adopted for D. On the one hand, treating the thickness from the first as infinitesimal (so that dy vanishes), and assuming on physical grounds that p is finite at the leading edge, we should suppress the first integral in (68) and so arrive at the formula

$$D = \mu \int \zeta \, dx. \quad \ldots \ldots \ldots \ldots \quad (76)$$

If, on the other hand, we were to discuss an elliptic cylinder, and from this proceed to the flat plate as the limiting case of unit eccentricity, the appropriate formula would be (75), and in this it is not obvious that the first integral would vanish as we proceeded to the limit, because $\partial\zeta/\partial\beta$ would attain infinite values at the edges.

A similar difficulty is presented in the relatively simple problem of irrotational motion. The formulæ for an elliptic cylinder make the leading edge a stagnation point, whatever be the eccentricity; and we do not arrive, in the limit, at the conclusion stated above (*cf.* § 9) in regard to the flat plate, that the fluid has uniform velocity in the direction Ox and no component velocity in the direction Oy. Instead, it can be shown that $\partial u/\partial x$ and $\partial v/\partial y$ attain large values at points immediately adjacent to the stagnation points.

Again, in the (approximate) solution obtained by BERRY and SWAIN* for the case of slow viscous flow past an elliptic cylinder lying with its major axis along the stream, examination shows that the integral $\int p dy$ does in fact tend to a finite limit as the length of the minor axis tends to zero.

Here we are concerned to test the accuracy of our methods of approximate calculation, and accordingly we shall adopt the formula (76), in order that our results may be comparable with those of BAIRSTOW, CAVE and LANG. They obtained the result†

$$D = \mu \int \zeta \, dx = 1 \cdot 527 \, \rho U^2 L \quad \ldots \ldots \ldots \ldots \quad (77)$$

(where L, as before, denotes the up-stream length of the plate‡) for the resistance, per unit length of the plate, at that value of REYNOLDS' number (namely, 4) which we have adopted in our investigation. For this value we have from (29)

$$UL = 4\nu = 4\mu\rho,$$

whence, substituting in (77), we obtain as an alternative form of their result

$$D = 4 \times 1 \cdot 527 \, \mu U, \quad \ldots \ldots \ldots \ldots \quad (78)$$

$$= 4 \times 1 \cdot 527 \, \mu, \quad \ldots \ldots \ldots \ldots \quad (79)$$

when the velocity U is given the value unity.

In our work we have taken this value for U, and by integration of the curve in fig. 9 we find that

$$\int \zeta \, dx = \int \zeta \, d\alpha = 4 \times 1 \cdot 563.$$

Hence, adopting the formula (76), we may write

$$D = \mu \int \zeta \, dx = 4 \times 1 \cdot 563 \mu, \quad \ldots \ldots \ldots \ldots \quad (80)$$

for comparison with (79). Moreover, we have shown that our investigations have the range predicted by the theory of dynamical similarity; so, generalizing equation (80), we may write either

$$\left.\begin{array}{ll} D = 4 \times 1 \cdot 563 \, \mu U, & \text{for comparison with (78),} \\ \text{or} \quad D = 1 \cdot 563 \, \rho U^2 L, & \text{for comparison with (77).} \end{array}\right\} \quad \ldots \ldots \quad (81)$$

Thus our calculations have given an estimate of resistance higher by $2\frac{1}{2}$ per cent. than that of the paper cited.

* 'Proc. Roy. Soc.,' A, vol. 102, p. 766 (1923).

† *Loc. cit.*, p. 415, equation F (21).

‡ This quantity was denoted by d in the paper cited.

Resistance of the Circular Cylinder.

32. In relation to the circular cylinder no uncertainty exists regarding the applicability of equation (75). In order to evaluate the first integral in that equation, we must have values of $\partial\zeta/\partial\beta$ for a number of points lying in the range defined by (60); given these, and having multiplied them by the corresponding values of y, we can effect the integration graphically or by SIMPSON'S rule.

Our approximate solution has been obtained by superposing "standard" solutions of a type defined by (62). Now corresponding with (39), which gives the value at any point (α, β) of the vorticity involved by a distribution $\sigma(\alpha')$ of doublet intensity, we have the expression

$$\left(\frac{\partial\zeta}{\partial\beta}\right)_{\alpha,\,0} = \int_{-\alpha_1}^{\alpha_1} \sigma(\alpha') \,.\, Z_\beta\,\{(\alpha - \alpha'),\, 0\}\, d\alpha'$$

for the value assumed by $\partial\zeta/\partial\beta$ at a point $(\alpha, 0)$. Here $Z_\beta(\alpha, \beta)$ stands for that function of α, β which represents $\partial\zeta_2/\partial\beta$, when ζ_2 has the meaning given in (36).

The integration, as before, is easily performed by graphical or approximate numerical methods. In this way we can deduce, for each of our standard solutions, values of $\partial\zeta/\partial\beta$ at a number of points in the range (60); and these being known we can, using the constants given in (66), deduce for the same points the values assumed by $\partial\zeta/\partial\beta$ in our approximate solution.

Table II summarizes the results of calculations which we have made on these lines. Our final integration gave the value

$$-\int y\,\frac{\partial\zeta}{\partial\beta}\,d\alpha = 3\cdot 80\ \mathrm{U}.$$

The evaluation of the second integral in (75) is a less laborious matter, and does not call for explanation. Our calculations gave the value*

$$\int \zeta dx = 3\cdot 85\ \mathrm{U},$$

whence we have from (75), as the resistance of the circular cylinder per unit length,

$$\mathrm{D} = \mu\left(-\int y\,\frac{\partial\zeta}{\partial\beta}\,d\alpha + \int \zeta dx\right)$$
$$= 7\cdot 65\ \mu\mathrm{U}. \quad \ldots\ldots\ldots \quad (82)$$

* It will be observed that the two integrals in (75) make almost equal contributions to the resistance. This might have been expected, since LAMB, working with OSEEN'S equation, has obtained a solution which makes these quantities identical; for small values of REYNOLDS' number the effect of substituting our approximation for OSEEN'S is not likely to be large.

TABLE II.

VALUES OF $\partial\zeta/\partial\beta$ IN THE RANGE $(-1 < \alpha < 1,\ \beta = 0)$.

$\alpha =$	$-1\cdot0$	$-0\cdot9$	$-0\cdot8$	$-0\cdot6$	$-0\cdot4$	$-0\cdot2$	0	$0\cdot2$	$0\cdot4$	$0\cdot6$	$0\cdot8$	$0\cdot9$	$1\cdot0$
First Component Solution $\sigma(\alpha) = (1 - \alpha^2)$	$+\infty$*	$3\cdot09$	$1\cdot17$	$-1\cdot07$	$-2\cdot69$	$-3\cdot86$	$-4\cdot79$	$-5\cdot23$	$-5\cdot22$	$-4\cdot53$	$-2\cdot77$	$-0\cdot61$	$+\infty$*
Second Component Solution $\sigma(\alpha) = \alpha(1 - \alpha^2)$	$-\infty$	$0\cdot24$	$2\cdot29$	$3\cdot69$	$3\cdot58$	$2\cdot63$	$1\cdot07$	$-0\cdot67$	$-2\cdot41$	$-3\cdot48$	$-3\cdot29$	$-1\cdot74$	$+\infty$
Third Component Solution $\sigma(\alpha) = \alpha^2(1 - \alpha^2)$	$+\infty$	$-0\cdot95$	$-2\cdot27$	$-2\cdot15$	$-0\cdot81$	$+0\cdot53$	$+1\cdot24$	$+0\cdot96$	$-0\cdot19$	$-2\cdot02$	$-2\cdot99$	$-2\cdot12$	$+\infty$
Approximate Solution for the Circular Cylinder	$+\infty$	$2\cdot62$	$1\cdot31$	$1\cdot40$	$1\cdot38$	$0\cdot72$	$-1\cdot08$	$-3\cdot66$	$-6\cdot75$	$-9\cdot23$	$-8\cdot66$	$-4\cdot46$	$+\infty$

* The infinite values attained by $\partial\zeta/\partial\beta$ when $\alpha = \pm 1$ are logarithmic, and cause no difficulty in the evaluation of $\int y\,(\partial\zeta/\partial\beta)\,d\alpha$, since y and the integrand $y(\partial\zeta/\partial\beta)$ vanish at these points.

In our calculations REYNOLDS' number has the value

$$R = \frac{Ud}{\nu} = 2, \quad \ldots \ldots \ldots \ldots \quad (59)\ bis$$

and we have seen that the results have the range predicted by theory of dynamical similarity.* Accordingly we may write (82) in the alternative form

$$D = 3\cdot825\ \rho U^2 d, \quad \ldots \ldots \ldots \ldots \quad (83)$$

which, like (82), applies to all cases in which R = 2.

LAMB,† working with OSEEN's approximation, has obtained a solution which is subject to a mathematical limitation additional to what is implied in that equation. His formula for the resistance is

$$D = \frac{4\pi\mu U}{\frac{1}{2} - \gamma - \log\frac{R}{8}},$$

reducing in our particular case (R = 2) to

$$D = 9\cdot60\mu U. \quad \ldots \ldots \ldots \ldots \quad (84)$$

In the paper by BAIRSTOW and others the limitation of LAMB's solution is removed. No general formula is obtained for the resistance, but in Table II of the paper values are given from which, by interpolation, we have obtained the formula

$$D = 8\cdot24\mu U \quad \ldots \ldots \ldots \ldots \quad (85)$$

in the case where R = 2.

Comparing our formula (82) with (84) and (85), we see that our investigation has led to an estimate of resistance which is less by 20 per cent. than LAMB's and by 7 per cent. than BAIRSTOW's. Since the general trend of experimental results is lower still,‡ some indication has been found that our approximation gives a better account than OSEEN's of the inertia terms in the governing equations.

Accuracy of our Approximate Equations.

33. In regard to the accuracy with which a solution of our modified equations (72) will approximate to a solution of the exact equations (2), very little can be said. Following RAYLEIGH,§ we may calculate the "constraining forces" which would be

* § 21.

† *Loc. cit.*, § 343.

‡ *Cf.* fig. 3 of a paper by EISNER in 'Proceedings of the Third International Congress for Applied Mechanics,' Stockholm, 1930; also WIESELSBERGER, 'Phys. Z.,' vol. 22, p. 321 (1921); and fig. 3 of the paper by BAIRSTOW and others.

§ 'Phil. Mag.' (5), vol. 36, p. 354 (1893). *Cf.* LAMB, *loc. cit.*, § 340.

required to make the solution exact; denoting these forces by X and Y we obtain, on subtraction of (72) from (2),

$$\left.\begin{aligned} X + \tfrac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \zeta\frac{\partial \alpha}{\partial y}, \\ Y + \tfrac{1}{2}\frac{\partial}{\partial y}(u^2+v^2) &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \zeta\frac{\partial \alpha}{\partial x}, \end{aligned}\right\}$$

or

$$\left.\begin{aligned} -X &= \zeta\left(v + \frac{\partial \alpha}{\partial y}\right), \\ Y &= \zeta\left(u + \frac{\partial \alpha}{\partial x}\right), \end{aligned}\right\} \qquad \qquad (86)$$

where u, v, ζ are values calculated from our approximate equation. We see from (86) that the forces will be of greatest importance close to the boundary of the cylinder, where $u = v = 0$; at a great distance from the cylinder, where equations (13) are satisfied very closely, they are unimportant. They are not conservative; for we have from (86)

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = \left(u + \frac{\partial \alpha}{\partial x}\right)\frac{\partial \zeta}{\partial x} + \left(v + \frac{\partial \alpha}{\partial y}\right)\frac{\partial \zeta}{\partial y}.$$

We desire to express our thanks to the Goldsmiths' Company, who by a studentship awarded to one of us have made this investigation possible. Also to Mr. J. H. Lavery, for collaboration in the graphical work and computations, and to the Secretary of the Aeronautical Research Committee and his staff, for assistance in the preparation of diagrams for reproduction.

Summary.

The problem considered is the steady motion in two dimensions, past a fixed cylindrical body, of an incompressible fluid possessing finite viscosity. The exact equations of motion are intractable, and Oseen has proposed to modify them in a way which has resulted in approximate solutions for certain shapes of cylinder. In this paper an alternative modification of the exact equations is proposed. The resulting equation for the vorticity has the same form as Oseen's in one particular case,—namely, when the cylinder is a flat plate presented edgeways to the fluid stream. In other cases the vorticity equation is altered, but it can be thrown into the form of Oseen's equation, provided that Cartesian co-ordinates are replaced by special co-ordinates appropriate to the particular shape of cylinder.

The modified equations are here solved by approximate methods (largely graphical) for two forms of cylinder: (*a*) the flat plate and (*b*) the circular cylinder. Our solution for (*a*) can be compared with the analytical solution of Bairstow, Cave and Lang, and provides a test of the accuracy of our approximate methods. Our solution for (*b*), since it is derived from a different equation, necessarily differs from that obtained by

LAMB on the basis of OSEEN's equation, and our estimate of resistance is 20 per cent. lower.

[(*Added in proof January* 27, 1933.) Since this paper was written it has come to our notice that the equation here proposed in substitution for (12) has been suggested previously by ZEILON (Appendix I to OSEEN's "Hydrodynamics," Leipzig, 1927), who refers to papers by BURGERS and others. In relation to the problem of heat conduction in a moving fluid (where an equation identical in form with (11) governs the temperature as determined by convection and conduction) BOUSSINESQ ('J. Math. pures appl.,' I, p. 285, 1905), has substituted "irrotational" expressions (13) for u and v, and by a change of variables has arrived at an equation similar to that numbered (22) in this paper. RUSSELL ('Phil. Mag.,' vol. 20, p. 591, 1910) and KING ('Phil. Trans.,' A, vol. 214, p. 373, 1914), have worked on similar lines. ZEILON's paper derives our approximation to the vorticity equation without entering into any detailed discussion; but BURGERS ('Akad. Wet. Amst.,' vol. 23, p. 952, 1921), has applied it to a discussion of flow at high values of REYNOLDS' number.]

APPENDIX.

It has been shown in § 15 that the problem of integrating equation (24) may be reduced (for the purpose of an approximate solution) to the problem of evaluating $\int r \sin\theta \, d\theta$ along a contour of given form, r and θ being polar co-ordinates which define a point on the contour.

In fig. 15, let C be the given contour, and P any point (r, θ). Then if we make

$$R = OQ = PN = r \sin\theta, \quad \ldots \ldots \ldots \ldots \quad \text{(i)}$$

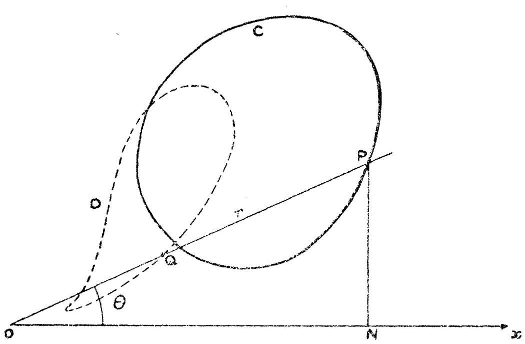


FIG. 15.

and proceed in the same way for a number of positions of P, we can construct a new contour relating R and θ. Our problem is now to evaluate the integral $\int R d\theta$ for the new contour (D in the diagram).

For this purpose the wheel of an ordinary Amsler planimeter has been mounted so that its carriage can slide on an arm which rotates about O. Since the plane of the wheel is always perpendicular to this arm, sliding the carriage along the arm has no effect on the reading, so long as the arm is kept stationary. If the arm is rotated about O, the rotation of the wheel (which is recorded by the counter mechanism) will be a measure of the quantity $\int \mathrm{R}d\theta$, where R is the distance from O of its point of contact with the paper. Thus we have only to make this point of contact trace out the contour D.

The tracing point of the planimeter is attached to the carriage at a definite distance (d) from the point of contact (Q) of the wheel and the paper, and can be arranged to lie always in the line OQ. Therefore in practice it is convenient to construct a contour relating $(\mathrm{R} + d)$ with θ, and to move the tracing point round this latter contour; it is evident that in this way Q will be made to follow the contour D.

Fig. 16, Plate 1, shows the actual instrument, of which the detailed design is largely due to Mr. S. Munday, senior mechanic in the Oxford University Engineering Laboratory. It has been tested on curves for which the integral could be calculated, and has been found to give very accurate results.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

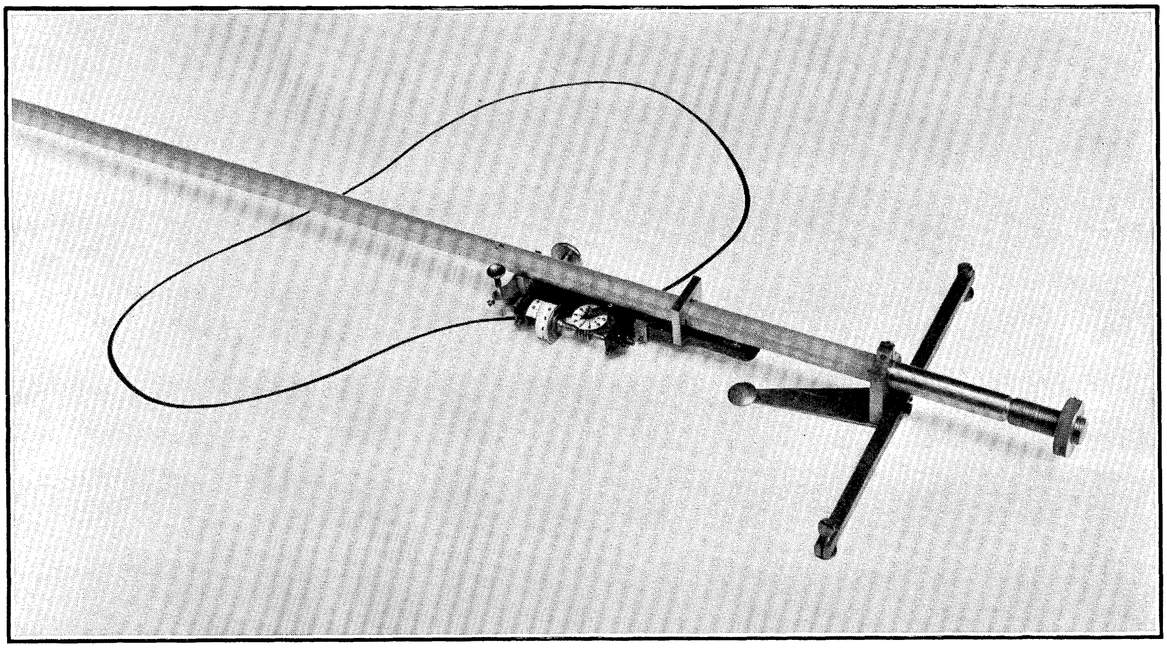


FIG. 16.

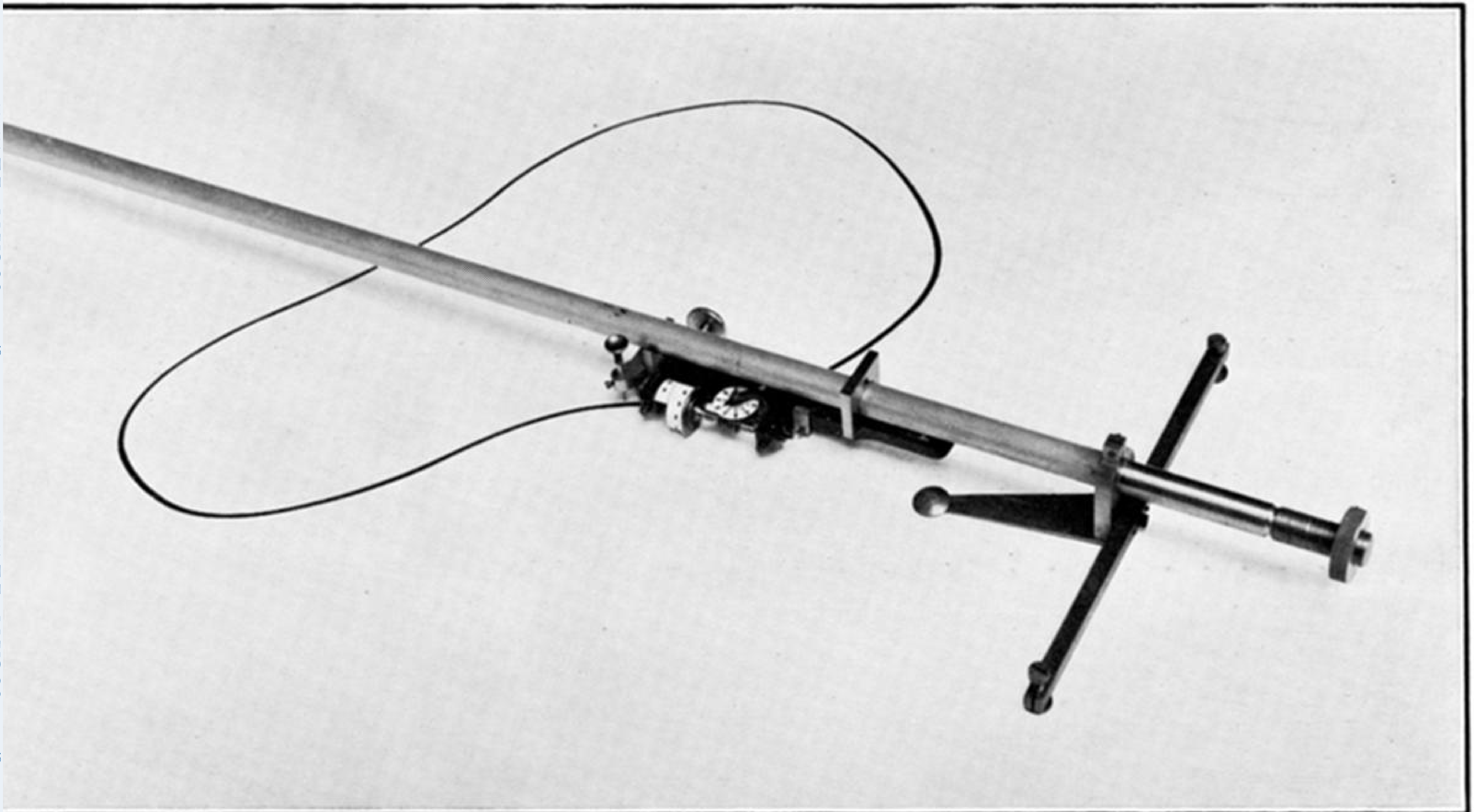


FIG. 16.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES